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IF THEY ACTUALLY..(U) OHIO STATE UNIV RESEARCH
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UNCLASSIFIED AFOSR-TR-85-0001 AFOSR-82-0307 F/G 12/1 NL

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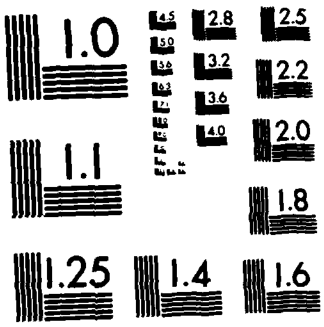

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RF Project 763265/714837
Annual Report

EFFECTS OF ASSUMING INDEPENDENT COMPONENT FAILURE TIMES, IF THEY ARE ACTUALLY DEPENDENT, IN A SERIES SYSTEM

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and

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For the Period
September 1, 1983 - September 30, 1984

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
Bolling Air Force Base, D.C. 20332

Grant No. AFOSR 82-0307

October 26, 1984

The Ohio State University
Research Foundation
1314 Kinnear Road
Columbus, Ohio 43212

85 02 [illegible] 001

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED

1b. RESTRICTIVE MARKINGS

2a. SECURITY CLASSIFICATION AUTHORITY

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE

3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S): Project 763265/714837 Annual Report

5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 85-0001

6a. NAME OF PERFORMING ORGANIZATION: Ohio State University Research Foundation

6b. OFFICE SYMBOL (If applicable)

7a. NAME OF MONITORING ORGANIZATION: Air Force Office of Scientific Research

6c. ADDRESS (City, State and ZIP Code): Department of Preventive Medicine and Department of Statistics, Columbus OH 43210

7b. ADDRESS (City, State and ZIP Code): Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332-6448

8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR

8b. OFFICE SYMBOL (If applicable): NM

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: AFOSR-82-0307

8c. ADDRESS (City, State and ZIP Code): Bolling AFB DC 20332-6448

10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2304	A5	

11. TITLE (Include Security Classification): EFFECTS OF ASSUMING INDEPENDENT COMPONENT FAILURE TIMES, IF THEY ARE ACTUALLY DEPENDENT, IN A SERIES SYSTEM, INTERIM SCIENTIFIC REPORT, GRANT AFOSR-82-0307, 1 SEPTEMBER 1983 - 30 /SEPTEMBER 1984.

12. PERSONAL AUTHOR(S): Melvin L. Moeschberger and John P. Klein

13a. TYPE OF REPORT: Interim

13b. TIME COVERED: FROM 1/9/83 TO 30/9/84

14. DATE OF REPORT (Yr., Mo., Day): 26 OCT 84

15. PAGE COUNT: 109

16. SUPPLEMENTARY NOTATION

17. COSATI CODES

FIELD	GROUP	SUB. GR.

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number): Modeling series systems; system reliability; competing risks; bivariate exponential distributions; test for independence; consistency of the product life estimator; dependent risks; estimation of component life.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The overall objective of this proposal is to investigate the robustness to departures from independence of methods currently in use in reliability studies when competing failure modes or competing causes of failure associated with a single mode are present in a series system. The first specific aim is to examine the error one makes in modeling a series system by a model which assumes statistically independent component lifetimes when in fact the component lifetimes follow some multivariate distribution. The second specific aim is to assess the effects of the independence assumption on the error in estimating component parameters from life tests on series systems. In both cases, estimates of such errors will be determined via mathematical analysis and computer simulations for several prominent multivariate distributions. A graphical display of the errors for representative distributions will be made available to researchers who wish to assess the possible erroneous assumption of independent competing risks. A third aim is to tighten the bounds on estimates of component reliability when the risks belong to a general dependence class of distributions (for example, positive quadrant dependence, positive regression (CONTINUED)

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐

21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL: Maj Brian W. Woodruff

22b. TELEPHONE NUMBER (Include Area Code): (202) 767- 5027

22c. OFFICE SYMBOL: NM

DD FORM 1473, 83 APR   EDITION OF 1 JAN 73 IS OBSOLETE.

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SECURITY CLASSIFICATION OF THIS PAGE

ITEM #20, ABSTRACT, CONTINUED: dependence, etc.). Major decisions involving reliability studies, based on competing risk methodology, have been made in the past and will continue to be made in the future. This study will provide the user of such techniques with a better understanding of the robustness of the analyses to departures from independent risks, an assumption commonly made by the methods currently in use.

Annual Report for the
Air Force Office of Scientific Research

by

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Title: Effects of Assuming Independent Component Failure Times, if They are Actually Dependent in a Series System

Period Covered: September 1, 1983 to September 30, 1984

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A. Table of Contents

B. TECHNICAL SECTION

I. Abstract

The overall objective of this proposal is to investigate the robustness to departures from independence of methods currently in use in reliability studies when competing failure modes or competing causes of failure associated with a single mode are present in a series system. The first specific aim is to examine the error one makes in modeling a series system by a model which assumes statistically independent component lifetimes when in fact the component lifetimes follow some multivariate distribution. The second specific aim is to assess the effects of the independence assumption on the error in estimating component parameters from life tests on series systems. In both cases, estimates of such errors will be determined via mathematical analysis and computer simulations for several prominent multivariate distributions. A graphical display of the errors for representative distributions will be made available to researchers who wish to assess the possible erroneous assumption of independent competing risks. A third aim is to tighten the bounds on estimates of component reliability when the risks belong to a general dependence class of distributions (for example, positive quadrant dependence, positive regression dependence, etc.). Major decisions involving reliability studies, based on competing risk methodology, have been made in the past and will continue to be made in the future. This study will provide the user of such techniques with a clearer understanding of the robustness of the analyses to departures from independent risks, an assumption commonly made by the methods currently in use.

II. Specific Objectives

The overall objective is to investigate the robustness to departures from independence to methods currently in use in reliability studies when competing failure modes or competing causes of failure associated with a single mode are present in a series system. We shall also refer to such competitive events as competing risks. The approach will be through the investigation of certain aspects of specific parametric multivariate distributions or by classes of distributions which are appropriate in reliability analyses when there are competing risks present.

The specific objectives are:

1) to assess the error incurred in modeling system life in a series system assumed to have independent component lifetimes when in fact the component lifetimes are dependent.

2) to assess the error in estimating component parameters (i.e., component reliability, mean component life, etc.) in a series system employing either parametric or nonparametric models which assume independent component failure times when in fact the lifetimes are dependent and follow some plausible multivariate distribution.*

3) to derive bounds on component reliability when the failure modes are dependent and fall in a particular dependence class (e.g., positive quadrant dependence, positive regression dependence, etc.).

4) to develop tests of independence, based on data collected from series systems, by making some restrictive assumption about the structure of the systems.**

* A plausible parametric multivariate distribution will be one that satisfies one of the following conditions:

i) the distribution of the minimum of the component failure times closely approximates widely accepted families of system life distributions.

or ii) the marginal distributions closely approximate the distributions of component failure times in the absence of other failure modes.

**This objective has been added to the original objectives because it answers a natural question raised by our preliminary investigation.

III. Introduction to Problem and Significance of Study

Alvin Weinberg (1978) in an editorial comment in the published proceedings of a workshop on Environmental Biological Hazards and Competing Risks noted that "the question of competing risks will not quietly go away: corrections for competing risks should be applied routinely to data." The problem of competing risks commonly arises in a wide range of experimental situations. Although we shall confine our attention in the following discussion to those situations involving series systems in which competing failure modes or competing causes of failure associated with a single mode are present, it is certainly true that we might just as easily speak of clinical trials, animal experiments, or other medical and biological studies where competing events interrupt our study of the main event of interest (cf. Lagokos (1979)).

Consider electronic or mechanical systems, such as satellite transmission equipment, computers, aircraft, missiles and other weaponry consisting of several components in series. Usually each component will have a random life length and the life of the entire system will end with the failure of the shortest lived component. We will examine two situations more closely in which competing risks play a vital role.

First, suppose we are attempting to evaluate system life from knowledge of the individual component lifetimes. Such an evaluation will utilize either an analysis involving mathematical statistics or a computer simulation. At a recent conference on Modeling and Simulation, McLean (1981) presented a scheme to simulate the life of a missile which consisted of many major components in series. The failure distribution associated with each component was assumed to be known (usually exponential or Weibull.) To arrive at the system failure distribution, the components were assumed to act independently of each other. Realistically, this may or may not be the case. If the component lifetimes were dependent for any reason, the computed system failure distribution (as well as its subsequent parameters such as system mean life and system reliability for a specified time) would only crudely approximate the true distribution. The first specific aim of this proposal is to ascertain the error incurred in modeling system life in a series system assumed to have independent component lifetimes (i.e., risks) when, in fact, the risks are dependent.

Second, suppose we wish to evaluate some aspect of the distribution of a particular failure mode based on a typical life test of a series system. The response of interest is the time until failure of a particular mode of interest. Frequently this response will not be observable due to the occurrence of some other event which precludes failure associated with the mode of interest. We shall term such competing events which interrupt our study of the main failure modes of interest as competing risks.

Competing risks arise in such reliability studies when

1) the study is terminated due to a lack of funds or the pre-determined period of observation has expired (Type I censoring).

2) the study is terminated due to a pre-determined number of failures of the particular failure mode of interest being observed (Type II censoring).

3) some systems fail because components other than the one of interest malfunction.

4) the component of interest fails from some cause other than the one of interest.

In all four situations, one may think of the main event of interest as being censored, i.e., not fully observable. In the first two situations, the time to occurrence of the event of interest should be independent of the censoring mechanism. In such instances, the methodology for estimating relevant reliability probabilities has received considerable attention (cf. David and Moeschberger (1978), Kalbfeish and Prentice (1980), Elandt-Johnson and Johnson (1980), Mann, Schafer, Singpurwalla (1974) and Barlow and Proschan (1975) for references and discussion). In the third situation, the time to failure of the component of interest may or may not be independent of the failure times of other components in the system. For example, there may be common environmental factors such as extreme temperature which may affect the lifetime of several components. Thus the question of dependent competing risks is raised. A similar observation may be made with respect to the fourth situation, viz., failure times associated with different failure modes of a single component may be dependent. For a very special type of dependence, the models discussed by Marshall-Olkin (1967), Langberg, Proschan and Quinzy (1978), and Langberg, Proschan, and Quinzy (1981) allow one to convert dependent models into independent ones.

If no assumptions whatever are made about the type of dependence between the distribution of potential failure times, there appears to be little hope of estimating relevant component parameters. In some situations, one may be appreciably misled (cf. Tsiatis (1975), Peterson (1976)). However, as Easterling (1980) so clearly points out in his review of Birnbaum's (1979) monograph

> "there seems to be a need for some robustness studies. How far might one be off, quantitatively, if his analysis is based on incorrect assumptions?"

The second specific aim will address this important issue. First if a specific parametric model which assumes independent risks has been used in the analysis, it would be of interest to know how the error in estimation is affected by this assumption of independence. That is, if independent specific parametric distributions are assumed for the failure times associated with different failure modes when we really should use a bivariate (or multivariate) distribution, then what is the magnitude of the error in estimating component parameters? Secondly, one may wish to allow for a less stringent type of model assumption, and ask the same question with regard to the estimation error. That is, if a nonparametric analysis is performed, assuming independent risks, when some types of dependencies may be present, then what is the magnitude of the estimation error?

The third specific aim will attempt to obtain bounds on the component reliability when the failure times belong to a broad dependence class (e.g., association, positive quadrant dependence, positive regression dependence, etc.). More details will be presented in the methods section.

In summary, competing risk analyses have been performed in the past and will continue to be performed in the future. This study will provide the user of such techniques with a clearer understanding of the robustness to departures from independent risks, an assumption which most of the methods currently in use assume.

IV. Progress Report on Second Year's Work

A summary of the first year's work is reported in the annual report dated October 26, 1983. The paper dealing with the Gumbel (1960) bivariate exponential model appeared in the August 1984 issue of Technometrics. A copy of this paper will be included in Appendix A. Also, the paper dealing with the asymptotic bias of the product limit estimator under dependent competing risks has been accepted for publication in the Journal of the Indian Association for Productivity, Reliability, and Quality Control. This article is to appear in 1984. (See Appendix C of the first year's annual report.) The paper on tests for independence with censored data has been revised and has now been accepted for publication in the Proceedings of the Conference on Reliability and Quality Control held at the University of Missouri in June 1984. The paper was presented as an invited paper at that meeting. A copy of the revised paper is included in Appendix B.

A paper which develops improved bounds on component reliability based on system data is displayed in Appendix C. These bounds, which are tighter than those of Peterson (1976), are obtained by specifying a range of possible concordances for the various modes of failure in a series system. This work was presented at the National Statistical Meetings in Philadelphia in August 1984.

Another paper, in a slightly different vein, which deals with assessing the goodness of several methods of estimating the survival function (reliability) when there is extreme right censoring, is displayed in Appendix D. This work, which was presented at the Spring statistical meeting in 1984 in Orlando, Florida, has been accepted for publication in Biometrics.

Another paper, which deals with reduced bias estimators of the joint reliability function for the Marshall-Olkin and Block-Basu bivariate exponential distribution, has been tentatively accepted for publication in Sankya. A copy of this paper appears in Appendix E.

Finally, work is near completion with respect to evaluating the consequences of erroneously assuming independence when modeling system reliability from complete component information for all three Gumbel bivariate exponential models, the Downton's bivariate exponential model, and Oakes bivariate model. It is anticipated that a paper will be written in the next month. This work will be presented in an invited talk at the Spring statistical meeting in Raleigh, North Carolina. Another paper will be written by January 1, 1985 which examines the impact of the independence assumption on the magnitude of the estimation error in estimating component reliability and mean life from data collected from series systems.

V. Methods

We refer to pages 8-52 of the original proposal for a discussion of the general methodology.

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APPENDIX A

CONSEQUENCES OF DEPARTURES FROM INDEPENDENCE IN EXPONENTIAL SERIES SYSTEMS

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Annual Report for Period 1 October 1983 - 30 September, 1984

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TECHNOMETRICS ©, VOL. 26, NO. 3, AUGUST 1984

Consequences of Departures From Independence in Exponential Series Systems

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This article investigates the consequences of departures from independence when the component lifetimes in a series system are exponentially distributed. Such departures are studied when the joint distribution is assumed to follow a Gumbel bivariate exponential model. Two distinct situations are considered. First, in theoretical modeling of series systems, when the distribution of the component lifetimes is assumed, one wishes to compute system reliability and mean system life. Second, errors in parametric and nonparametric estimation of component reliability and component mean life are studied based on life-test data collected on series systems when the assumption of independence is made erroneously. Systems with two components are studied.

KEY WORDS: Competing risks; Component life; Modeling series systems; Robustness studies; System reliability; Gumbel bivariate exponential.

1. INTRODUCTION

Consider a system consisting of several components linked in series. For such a system the failure of any one of the components causes the system to fail. Common assumptions made in modeling and analyzing data from such a system are that the component lifetimes are independent and exponentially distributed. Many authors have considered the problem of analyzing a series system with exponential component lives. For example, confidence bounds for system reliability assuming independent exponentially distributed component lifetimes were presented in Mann (1974) and Mann and Grubbs (1974). (See Mann, Schafer, and Singpurwalla 1974 for a more comprehensive review.) More recently, work invoking the assumption of independent exponentially distributed lifetimes has been presented by Chao (1981) and Miyamura (1982). Estimation of component parameters from series system data has been treated by Boardman and Kendell (1970) in the context of independent exponential component lives. Some authors suggest a nonparametric alternative to the estimation of component reliability based on series system data (compare Kalbfleisch and Prentice 1980 and Lawless 1982).

The assumption of independence is essential to these analyses and an important concern. Several authors have shown that this assumption, by itself, is not testable because based on data from a series system, there is no way to distinguish between an independent and a dependent model. (See Tsiatis 1975, Peterson 1976, and Basu 1981 for a discussion of nonidentifiability results.) In many situations one may be appreciably misled by the independence assumption.

Lagakos (1979), in a study of the effects of various types of dependence among component lifetimes, notes that most methods of analysis have assumed noninformative models of which independence is a special case. He points out, "it is important to be aware of the possible consequences of making this assumption when it is false" (p. 152). Furthermore, Easterling (1980) states in his review of Birnbaum's (1979) monograph on competing risks, "there seems to be a need for some robustness studies. How far might one be off, quantitatively, if his analysis is based on incorrect assumptions?" (p. 131).

In this article we consider the consequences of departures from independence when the component lifetimes are exponentially distributed. Such departures may be related to some common environmental factor that is present only when the components are linked together in series. The load each component is subject to is either reduced or increased according to the age of the system. To study such departures, we have selected a model proposed by Gumbel (1960). Gumbel's model retains the assumption of exponentially distributed component lifetimes while allowing the flexibility of both positive or negative mild correlation between component lifetimes.

The effects of a departure from the assumption of independent component lifetimes in a series system will be addressed for two distinct situations. The first situation arises in modeling the performance of a theoretical series system constructed from two components whose lifetimes are exponentially distributed. Here, based on testing each component separately or on engineering design principles, it is reasonable to assume that the components are exponentially distributed with known parameter values. Based on this information, we wish to calculate parameters such as the mean life or reliability of a series system constructed from these components. In Section 2 we describe how the values of these quantities are affected by departures from independence when the component parameters are completely specified. In Section 3 we study the performance of the Mann-Grubbs (1974) confidence bounds on system reliability for small sample sizes and for varying degrees of correlation, when the component parameters are estimated from component data.

The second situation involves making inferences about component lifetime distributions, reliabilities, and mean lives from data collected on series systems. Commonly, data collected on such systems are analyzed by assuming a constant-sum model, of which independence is a special case (compare Williams and Lagakos 1977 and Lagakos and Williams 1978). In Section 4 we study the properties of the maximum likelihood estimators of component parameters calculated under an assumption of independent exponential component lifetimes when the component lifetimes are Gumbel bivariate exponential. Because of the widespread use of nonparametric estimates of component reliability, we also present in Section 5 the estimation error of the Kaplan-Meier (1958) estimator when the assumption of independence is made erroneously.

2. MODELING SYSTEM RELIABILITY FROM COMPLETE COMPONENT INFORMATION

Consider a two-component series system with component life lengths X_1, X_2. Suppose that X_i has an exponential survival function

$$\bar{F}_i(t) = P(X_i > t) = \exp(-\lambda_i t),$$
$$\lambda_i, \quad t > 0, \quad i = 1, 2.$$

This assumption is made on the basis of extensive testing of each component separately or on knowledge of the underlying mechanism of failure. The value of λ_i is assumed known. If X_1, X_2 are independent, then the time to system failure has an exponential distribution with failure rate $\lambda = \lambda_1 + \lambda_2$, and the system reliability is given by

$$\bar{F}_I(t) = P[\min(X_1, X_2) > t \mid \text{independence}] = \exp(-\lambda t). \tag{2.1}$$

Suppose that the actual joint distribution of (X_1, X_2) has the form proposed by Gumbel (1960), namely,

$$P(X_1 > x_1, X_2 > x_2) = [\exp(-\lambda_1 x_1 - \lambda_2 x_2)] \times [1 + \alpha(1 - \exp(-\lambda_1 x_1))(1 - \exp(-\lambda_2 x_2))]. \tag{2.2}$$

The joint probability density of (X_1, X_2) is

$$f(x_1, x_2) = \lambda_1\lambda_2[\exp(-\lambda_1 x_1 - \lambda_2 x_2)] \times [1 + \alpha(2\exp(-\lambda_1 x_1) - 1) \times (2\exp(-\lambda_2 x_2) - 1)], \tag{2.3}$$

where in both (2.2) and (2.3), x_1, x_2, λ_1, $\lambda_2 > 0$, $-1 \le \alpha \le 1$. This distribution has marginal survival functions equivalent to those for the independent model, which, in part, is the reason for choosing it. The correlation between X_1, X_2 is $\rho = \alpha/4$, and $\alpha = 0$ is equivalent to X_1, X_2 being independent. For $\rho > 0$ (<0) the components are positively (negatively) quadrant-dependent (see Barlow and Proschan 1975). Furthermore, the conditional expectation of X_1, given $X_2 = x_2$, is

$$E(X_1 \mid X_2 = x_2) = \frac{1}{\lambda_1}[1 + 2\rho - 4\rho\exp(-\lambda_2 x_2)].$$

If (X_1, X_2) have the joint distribution (2.3), then the true system reliability is

$$\bar{F}_D(t) = P[\min(X_1, X_2) > t \mid \text{dependence}] = \exp(-\lambda t)[1 + 4\rho(1 - \exp(-\lambda_1 t)) \times (1 - \exp(-\lambda_2 t))]. \tag{2.4}$$

From (2.1) and (2.4) we see that the error in modeling system reliability is

$$\Delta(t) = \bar{F}_D(t) - \bar{F}_I(t) = 4\rho[1 - \exp(-\lambda_1 t)][1 - \exp(-\lambda_2 t)] \times \exp(-(\lambda_1 + \lambda_2)t). \tag{2.5}$$

Note that $|\Delta(t)|$ increases as $|\rho|$ increases, for fixed λ_1, λ_2, and t. The magnitude of $\Delta(t)$, of course, depends on λ_1, λ_2, t, and ρ. When $\lambda_1 = \lambda_2 = \phi$, one can show that $\Delta(t)$ is maximized at $t = [\ln 2]/\phi$ (fixing ρ and ϕ). The value of $|\Delta(t)|$ at this point is $|\rho|/4$, which is at most 1/16. Representative values of $\bar{F}_D(t)$ for $\lambda_1 = 1$, $\lambda_2 = 1.5$, and $\rho = -.25$, $-.125$, 0, .125, and .25 are plotted in Figure 1. The curve with $\rho = 0$ corresponds to the system reliability if the assumption of independence is true. Since most applications of interest involve reliabilities of .75 or greater, in Figure 2 we plot the ratio of the 100 pth upper percentiles under dependence and independence versus the correlation. From Figure 2 it appears that when the predicted system reliability under independence is greater than .90, misspecifying the dependence parameter has little effect. In the range where the predicted system reliability under independence is less than .75, however, misspecifying the de-

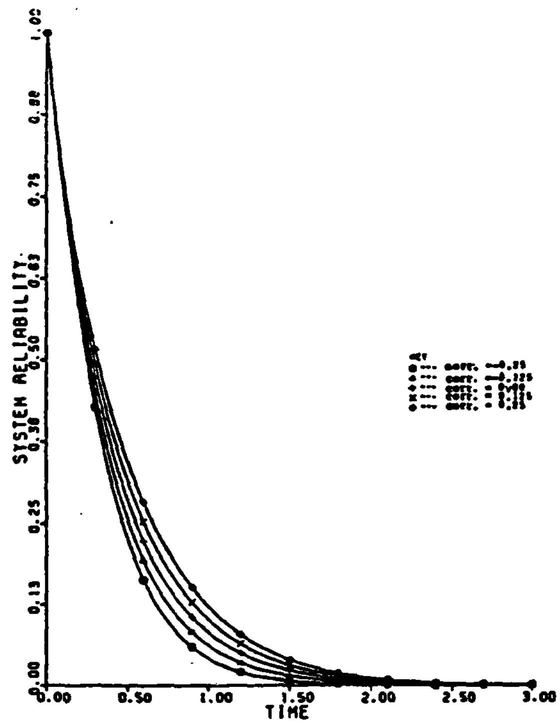

Figure 1. *System Reliability for Gumbel's Model*, $\lambda_1 = 1, \lambda_2 = 1.5$.

pendence parameter may lead to errors exceeding 6%. Maximum values of $|\Delta(t)|$ are presented in Table 1 for $\lambda_1 = 1$ and various values of λ_2.

The mean time to system failure based on (2.1), assuming independence, is

$$\mu_I = 1/(\lambda_1 + \lambda_2), \qquad (2.6)$$

and that based on (2.4) is

$$\mu_D = \frac{1}{(\lambda_1 + \lambda_2)} + 4\rho\left[\frac{3}{2(\lambda_1 + \lambda_2)} - \frac{1}{(2\lambda_1 + \lambda_2)} - \frac{1}{(\lambda_1 + 2\lambda_2)}\right]. \qquad (2.7)$$

The amount of error in modeling system mean life is

$$\mu_D - \mu_I = \frac{6\rho\lambda_1\lambda_2}{(\lambda_1 + \lambda_2)(2\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2)} = \frac{6\rho\lambda_1\lambda_2\mu_I}{(2\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2)}, \qquad (2.8)$$

whose absolute value obviously increases as $|\rho|$ increases. If $\lambda_1 = \lambda_2$, this error reduces to $2\rho\mu_I/3$, which has a maximum absolute value of $\mu_I/6$.

It is apparent from Table 1 and Equations (2.5) and (2.8) that the error in modeling system reliability and mean system life, based on independence, increases as $|\rho|$ increases and is a function of the relative sizes of λ_1 and λ_2. In particular, when the mean life of one component is substantially greater than the mean life of the second component, then the behavior of the system is well approximated by the behavior of the shorter-lived component acting alone. This can be seen in (2.4) and (2.7) by letting $\lambda_1 \to 0$. In this instance we also see, from (2.5) and (2.8), that the amount of error incurred by assuming independence is negligible.

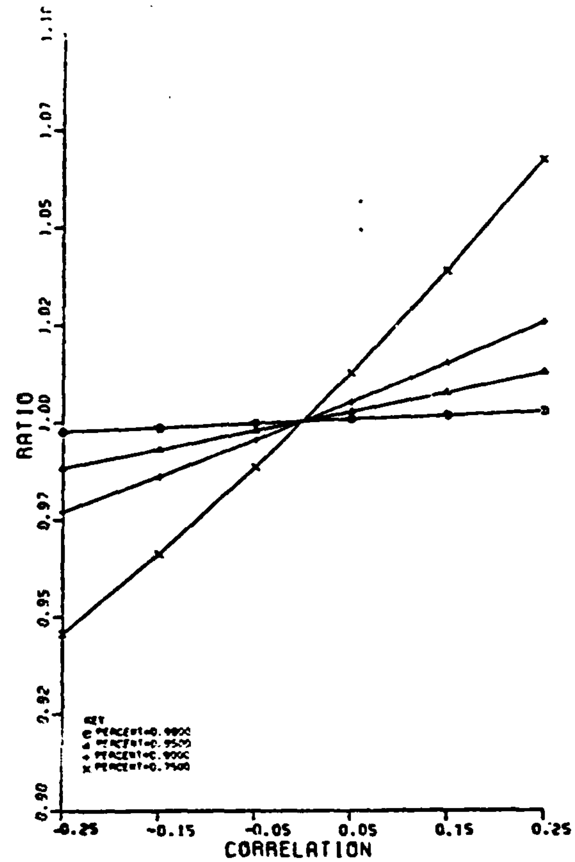

Figure 2. *Ratio of 100 pth Percentile Under Dependence and Independence Versus Correlation for* $\lambda_1 = 1, \lambda_2 = 1.5$.

3. ESTIMATING SYSTEM RELIABILITY FROM COMPONENT DATA

A common practice in predicting system reliability is to test each of the components independently and then to use the data to obtain confidence bounds on

Table 1. *Maximum Values of* $|\Delta(t)|$ *for* $\lambda_1 = 1$ *and Various Values of* λ_2

λ_2	Max $\lvert\Delta(t)\rvert$
2	.056
4	.041
8	.025
16	.014

system reliability. These bounds, obtained by Mann and Grubbs (1974), assume that the component lifetimes are exponential and that the components act independently when linked in series. In the bivariate case the bounds are computed as follows: For the jth component, suppose that n_j prototypes have been tested until $r_j (\leq n_j)$ failures occur. Let Z_j be the total time on test for the jth component. Define

$$M^* = \sum (r_j - 1)/Z_j + \frac{\sum (r_j - 1)/Z_j^3}{\sum (r_j - 1)/Z_j^2}, \qquad (3.1)$$

and

$$V^* = \sum (r_j - 1)/Z_j^2 + \frac{\sum (r_j - 1)/Z_j^4}{\sum (r_j - 1)/Z_j^2}. \qquad (3.2)$$

An approximate γ-level lower confidence bound for system reliability at time t_m is

$$\exp\left[-t_m M^* \{1 - V^*/(9M^{*2}) + \eta_\gamma (V^*)^{1/2}/(3M^*)\}^3\right], \qquad (3.3)$$

where η_γ is the 100γ percentile of a standard normal random variable.

When the system being evaluated has dependent components, these bounds may be misleading. The problem is that component data are independent, since the components are tested separately, but when they are put together into a system, some interdependence may develop. Of course, such dependence is not detectable in the absence of some system data, since the data on components we see are independent. To study the performance of the bound (3.3) when the correct system model is the Gumbel model (2.2), a simulation study was performed. For each simulated sample, n_j observations from exponential populations with mean $1/\lambda_j$, $j = 1, 2$, were simulated. The two samples were generated independently. The confidence bound (3.3) was obtained. This was then compared to the true system reliability at various ρ's obtained from (2.4). Ten thousand such bounds were simulated for each set of parameter values. The estimated coverage probabilities for the Mann-Grubbs bounds (i.e., the proportion of times that the Mann-Grubbs intervals assuming independence actually contained the true system reliability) for $n_1 = n_2 = 3$, 5, 10, $\lambda_1 = 1.0$, $\lambda_2 = 1, 5$, at $t_m = .1$ are reported in Table 2. Here the true system reliability under dependence ranges from .7684 at $\rho = -.25$ to .7891 at $\rho = .25$, with a value of .7788 at $\rho = .0$.

The results in Table 2 show that at high negative correlations, the coverage probabilities are significantly lower than claimed under independence, and for a high positive correlation, the intervals are conservative. This trend becomes more exaggerated as n_1, n_2 increase because the bound approaches the reliability under independence. As seen in Section 2, the true

Table 2 *Estimated Coverage Probabilities for Mann - Grubbs Bounds* $(-\lambda_1 = 1.0, \lambda_2 = 1.5)$

			Correlation						
n_1	n_2	γ	−.25	−.15	−.05	0	.05	.15	.25
3	3	.95	93.41[b]	94.11[b]	94.74	95.05	95.27	95.80[a]	96.22[a]
3	3	.90	87.40[b]	88.42[b]	89.32[b]	89.78	90.20	91.18[a]	92.15[a]
3	3	.75	71.03[b]	72.53[b]	74.13[b]	74.88	76.58	77.34[a]	78.81[a]
5	5	.95	93.19[b]	94.04[b]	94.90	95.26	95.62[a]	96.17[a]	96.81[a]
5	5	.90	87.12[b]	88.48[b]	89.85	90.39	91.10	92.13[a]	93.14[a]
5	5	.75	69.68[b]	72.02[b]	74.10[b]	75.13	76.14[a]	78.32[a]	80.03[a]
10	10	.95	92.03[b]	93.42[b]	94.58	95.08	95.51[a]	96.42[a]	97.14[a]
10	10	.90	85.93[b]	87.70[b]	89.34[b]	90.21	91.05[a]	92.56[a]	93.93[a]
10	10	.75	67.77[b]	70.90[b]	74.12[b]	75.63	77.05[a]	79.87[a]	82.56[a]

[a] At least two standard errors above specified level.
[b] At least two standard errors below specified level.
NOTE: Standard errors of the above estimates are approximately .2 for the .95 level, .3 for the .90 level, and .4 for the .75 level.

reliability at t is an increasing function of ρ so that asymptotically coverage probabilities approach 0 (or 1) for $\rho < 0$ (>0). For sample sizes in the range of 3 to 10, the estimated coverage probabilities for $\rho < 0$ are statistically significantly lower than expected. On the practical side, however, they are not of sufficient magnitude to cause great concern, especially at $\gamma = .95$.

4. PARAMETRIC ESTIMATION OF COMPONENT PARAMETERS

In this section we are interested in examining how the independence assumption affects the magnitude of the estimation error in estimating component mean life from data collected on series systems. That is, for each system tested, we observe its failure time and an indicator variable that tells us which component caused the system to fail. We are interested in how varying degrees of dependence affect the bias and mean squared error (MSE) of the maximum likelihood estimator of component mean life obtained by assuming independent component lifetimes.

We assume that the two components' survival functions are $\bar{F}_i(t) = \exp(-\lambda_i t)$, $i = 1, 2$, and a life test is conducted by putting n systems on test. We observe n_i systems failing because of failure of the ith component, $i = 1, 2$. Let T denote the sum of all n failure times. From Moeschberger and David (1971), the maximum likelihood estimator of λ_i, assuming independence, is

$$\hat{\lambda}_i = n_i/T, \qquad i = 1, 2,$$

so the estimator of component mean life, $\mu_i = \lambda_i^{-1}$, is

$$\hat{\mu}_i = T/n_i \quad \text{if} \quad n_i > 0. \qquad (4.1)$$

Now suppose that we are in fact sampling from the Gumbel distribution (2.3). For this model, component mean life is the same as in the independent case. The random variables (n_i, T) are independent (the conditional distribution of T given n_i is free of n_i), and n_i is binomial with parameters n and $p_i = P(\min(X_1, X_2) = X_i)$. For this model,

$$p_1 = P(X_1 < X_2)$$

$$= \lambda_1 \left\{ \frac{1}{\lambda_1 + \lambda_2} + \frac{4\rho(\lambda_1 - \lambda_2)\lambda_2}{(\lambda_1 + \lambda_2)(2\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2)} \right\}, \tag{4.2}$$

with $p_2 = 1 - p_1$. From Mendenhall and Lehman (1960), approximations to the moments of $1/n_i$, conditional on $n_i > 0$, are

$$E(1/n_i \mid n_i > 0) = \frac{n-2}{n(a-1)} \tag{4.3}$$

and

$$E(1/n_i^2 \mid n_i > 0) = \frac{(n-2)(n-3)}{n^2(a-1)(a-2)}, \tag{4.4}$$

where $a = (n-1)p_i$. The expected value of T is given by $n\mu_D$, where μ_D is given by (2.7), and

$$E(T^2) = n\left[\frac{2 + 10\rho}{(\lambda_1 + \lambda_2)^2} - 8\rho\left(\frac{1}{(2\lambda_1 + \lambda_2)^2} + \frac{1}{(\lambda_1 + 2\lambda_2)^2}\right)\right] + n(n-1)\mu_D^2. \tag{4.5}$$

Thus, the bias and MSE of $\hat{\mu}_i$, conditional on $n_i > 0$, under this model are

$$B(\hat{\mu}_i) = E(\hat{\mu}_i - \mu_i) = \frac{(n-2)\mu_D}{[(n-1)(p_i - 1)]} - \mu_i, \tag{4.6}$$

and

$$\text{MSE}(\hat{\mu}_i) = E(T^2)E(1/n_i^2 \mid n_i > 0) - \frac{2\mu_i(n-2)\mu_D}{[(n-1)p_i - 1]} + \mu_i^2, \tag{4.7}$$

for $i = 1, 2$.

We note that for large samples,

$$\lim_{n \to \infty} B(\hat{\mu}_i) = \frac{\mu_D}{p_i} - \mu_i \tag{4.8}$$

$$\lim_{n \to \infty} \text{MSE}(\hat{\mu}_i) = \left(\lim_{n \to \infty} B(\hat{\mu}_i)\right)^2 \tag{4.9}$$

for $i = 1, 2$. For $\lambda_1 = \lambda_2$ from (4.6), we see that

$$B(\hat{\mu}_1) = \frac{1 + 2(n-2)\rho/3}{(n-3)} \mu_1$$

$$= \frac{\mu_1}{n-3} + \frac{2(n-2)\rho\mu_1}{3(n-3)}. \tag{4.10}$$

A similar expression holds for $B(\hat{\mu}_2)$. Note that (4.10) consists of two terms. The first term, reflecting sampling error, is positive for all n and dominates the bias expression for small n. The second term, reflecting modeling error, takes on the same sign as the correlation and dominates for large n, approaching the limit of $2\rho\mu_1/3$.

When $\lambda_1 = \lambda_2$,

$$\text{MSE}(\hat{\mu}_1) = \frac{2\mu_1^2(n^2 - 2n + 3)}{n(n-5)(n-3)} + \frac{2\mu_1^2(n-2)}{9n(n-5)(n-3)} \times \{(19n - 21)\rho + 2(n-3)(n-1)\rho^2\}. \tag{4.11}$$

As in the bias expression, the MSE reflects a sampling error term and a modeling error term. The modeling error is a quadratic function of ρ for fixed n. For $n > 5$, this error is increasing in ρ for

$$\rho > -\frac{1}{4}\frac{(19n - 21)}{(n-3)(n-1)}$$

and decreasing in ρ for

$$\rho < -\frac{1}{4}\frac{(19n - 21)}{(n-2)(n-1)}.$$

For sample sizes between 5 and 21, the modeling error, and hence the MSE, is a strictly increasing function for all $\rho \in [-\frac{1}{4}, \frac{1}{4}]$. For $n > 21$, the minimum MSE is achieved at $\rho < 0$. As n approaches ∞, the value at which the smallest MSE occurs tends to 0.

For unequal component means a similar result holds. Figure 3 shows the bias as a function of ρ for

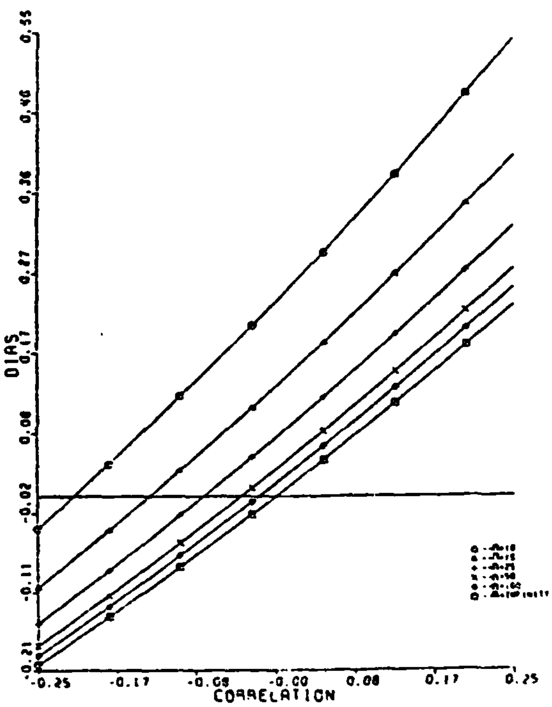

Figure 3. *Bias of $\hat{\mu}_1$ Under Gumbel's Model for $\lambda_1 = 1, \lambda_2 = 1.5$.*

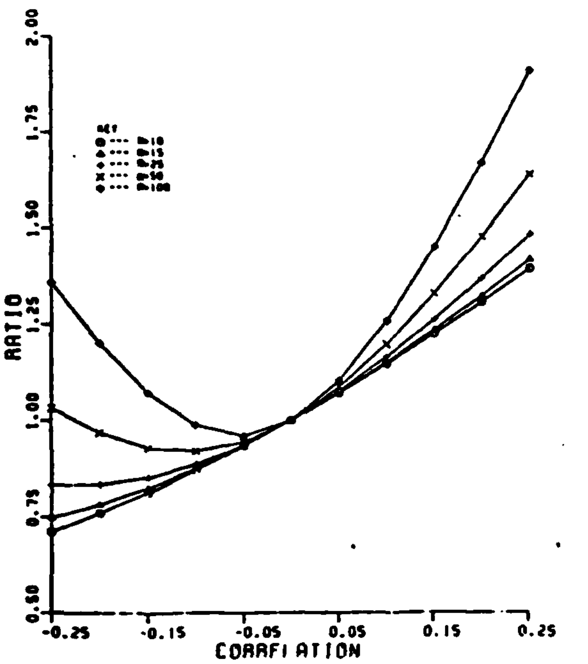

Figure 4. *Ratio of* $\sqrt{MSE(\hat{\mu}_1 \mid \rho)/MSE(\hat{\mu}_1 \mid \rho = 0)}$ *for Various Sample Sizes n and for* $\lambda_1 = 1, \lambda_2 = 1.5$.

various sample sizes when $\lambda_1 = 1.0$ and $\lambda_2 = 1.5$. Figure 4 depicts the ratio

$$\sqrt{\text{MSE}(\hat{\mu}_1 \mid \rho)/\text{MSE}(\hat{\mu}_1 \mid \rho = 0)}$$

as a function of ρ for various sample sizes when $\lambda_1 = 1, \lambda_2 = 1.5$.

5. BIAS OF THE PRODUCT LIMIT ESTIMATOR

A second approach to the problem of estimating component parameters is via the nonparametric estimator proposed by Kaplan and Meier (1958). Investigators who routinely use nonparametric techniques may take this approach in hopes of obtaining estimators that are robust with respect to the assumption of exponentiality. The purpose of this section is to show that such estimators are not necessarily robust with respect to the assumption of independence when the marginals are, in fact, exponential.

The product limit estimator, assuming independent risks, is constructed as follows. As before, suppose n systems are put on test at time 0 and n_i systems fail owing to failure of component i. Let $X_{i(1)}, \ldots, X_{i(n_i)}$ denote the ordered times at which these n_i events occur, and let $r_{i1}, \ldots, r_{in_i}$ be the ranks of those ordered survival times among all n ordered lifetimes. The component reliability for the ith component at time x may now be estimated by the product of the individual conditional survival probabilities, namely, by

$$\hat{\bar{F}}_i(x) = 1 \quad \text{if} \quad x < x_{i(1)}$$
$$= \prod_{j=1}^{j(i,x)} \frac{n - r_{ij}}{n - r_{ij} + 1}, \qquad x > x_{i(1)},$$

where $j(i, x)$ is the largest value of j for which $x_{i(j)} < x$. A special note is needed to cover the case in which $x_{i(n_i)}$ is not the largest observed death. To avoid this problem, we shall define $\hat{\bar{F}}_i(x) = 0$ for x greater than the largest observed failure time.

If the component lifetimes in fact follow the Gumbel bivariate exponential, we can see that the Kaplan-Meier estimator is not consistent. For $i = 1$, the Kaplan-Meier estimator is not estimating $\bar{F}_1(t)$, but, rather, another survival function, $\bar{H}_1(t)$, given by

$$\bar{H}_1(t) = \exp\left\{-\lambda_1 \int_0^t \frac{[1 + 4\rho(1 - e^{-\lambda_2 u})(1 - 2e^{-\lambda_1 u})]}{[1 + 4\rho(1 - e^{-\lambda_1 u})(1 - e^{-\lambda_2 u})]}\, du\right\}, \quad t > 0. \tag{5.1}$$

Note that if $\lambda_1 = \lambda_2 = \phi$, (5.1) is simplified to

$$\bar{H}_1(t) = e^{-\phi t}[1 + 4\rho(1 - e^{-\phi t})^2]^{1/2}, \tag{5.2}$$

which is increasing in ρ. Similarly, $\hat{\bar{F}}_2(t)$ is actually estimating $\bar{H}_2(t)$, which is defined analogously.

Measures of the error in estimating $\bar{F}_i(t)$ by $\hat{\bar{F}}_i(t)$ are

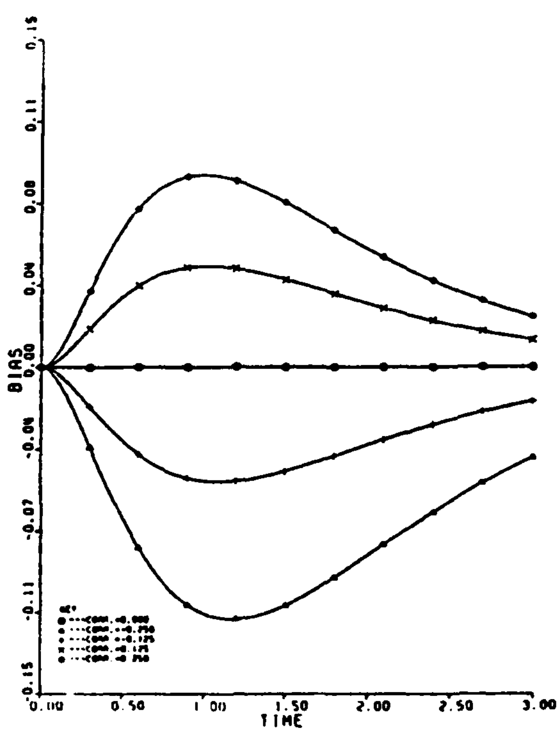

Figure 5. *Bias of Kaplan–Meier Estimate,* $\hat{\bar{F}}_1(t)$, $\lambda_1 = 1, \lambda_2 = 1.5$.

the bias and MSE of $\tilde{F}_i(t)$ computed under the dependence model. Under this model, the Kaplan-Meier estimator is equivalent to the estimator one would obtain based on n observations from an independent system with component survival distributions $\bar{H}_i$ given by (5.1) or, if $\lambda_1 = \lambda_2$, by (5.2). Hence from Kaplan and Meier (1958), the variance of $\tilde{F}_i(t)$ is given by

$$V(\tilde{F}_i(t)) = \bar{H}_i(t)^2 \int_0^t \frac{|d\bar{H}_i(u)|}{n\bar{H}_i(u)^2}. \tag{5.3}$$

Thus from (5.1) and (5.2), the bias and MSE of $\tilde{F}_i(t)$ are

$$B(\tilde{F}_i(t)) = \bar{H}_i(t) - \bar{F}_i(t), \qquad t \geq 0, \tag{5.4}$$

and

$$\text{MSE } (\tilde{F}_i(t)) = (\bar{H}_i(t) - \bar{F}_i(t))^2 + \bar{H}_i(t)^2 \int_0^t \frac{|d\bar{H}_i(u)|}{n\bar{H}_i(u)^2}, \qquad t > 0. \tag{5.5}$$

The estimator is not consistent, since $B(\tilde{F}_i(t))$ is independent of n and not necessarily zero. Also, MSE $(\tilde{F}_i(t))$ consists of a factor that depends only on the model error and is free of sample size and of a term that tends to 0 as n tends to infinity.

Note that in the case of equal component lifetime distributions, $\lambda_1 = \lambda_2 = \phi$, the bias determined from (5.2) and (5.4) simplifies to

$$B(\tilde{F}_i(t)) = e^{-\phi t}\{[1 + 4(1 - e^{-\phi t})^2]^{1/2} - 1\}. \tag{5.6}$$

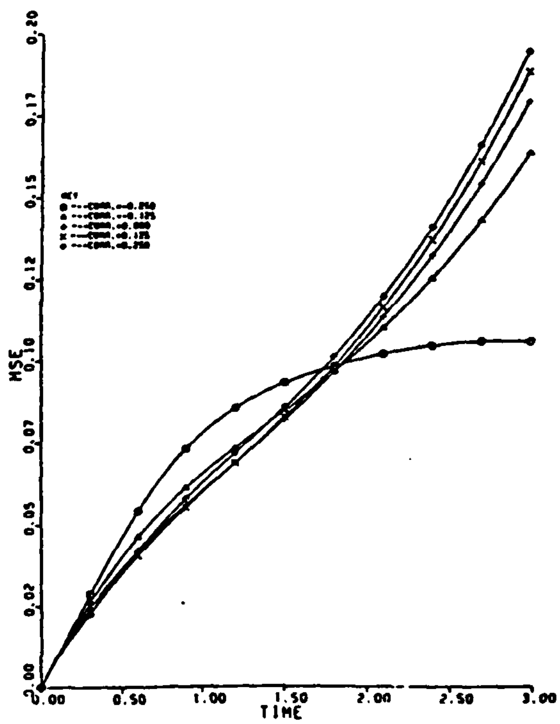

Figure 6. *MSE of Kaplan–Meier Estimate*, $\bar{F}_1(t)$, $\lambda_1 = 1, \lambda_2 = 1.5, n = 10$.

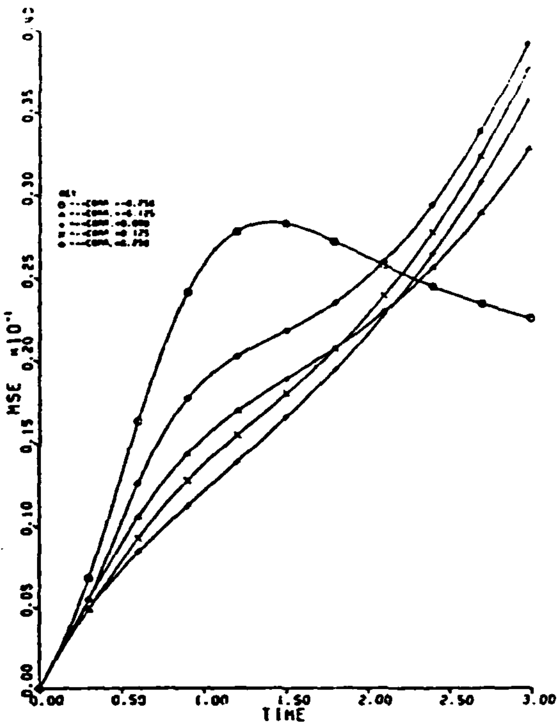

Figure 7. *MSE of Kaplan–Meier Estimate*, $\bar{F}_1(t)$, $\lambda_1 = 1, \lambda_2 = 1.5, n = 50$.

In the general case, the integral in (5.1) needs to be evaluated numerically. The bias of the Kaplan-Meier estimator was calculated for various values of λ_i and ρ. A representative plot of the bias appears in Figure 5, where $\lambda_1 = 1$, $\lambda_2 = 1.5$, and $|\rho| = 0, .125, .250$. It is apparent that the bias is largest for values of t in the neighborhood of an interval that captures the mean component lifetimes. The absolute value of the bias ranges from 0 to .11 in this example.

MSE $(\tilde{F}_i(t))$ was calculated for various values of λ_i, n, and ρ. Its magnitude is typified in Figures 6 and 7, where $\lambda_1 = 1$, $\lambda_2 = 1.5$, and $n = 10, 50$, respectively. For $\lambda_1 = 1$, $\lambda_2 = 1.5$, and $n = \infty$, MSE $(\tilde{F}(t))$ may be obtained by squaring (5.4) or by squaring the ordinate values in Figure 5. The MSE of the Kaplan-Meier estimator may be quite large for small sample size n and moderately large for "large" ρ, the former being a more crucial factor than the latter.

6. SUMMARY

The results presented here show that for the Gumbel model, one may be misled by falsely assuming *independence of component* lifetimes in a series system. In modeling system reliability based on complete information about two marginal component life distributions, effects of erroneously assuming independence of component lifetimes is most pronounced for system reliabilities smaller than .75. For system reliabilities larger than .90, this effect is too small to be of practical interest. The effects of a departure from independence on the Mann-Grubbs bounds for small sample sizes seems to be negligible for confidence levels greater than .90. But for either large sample sizes or smaller confidence levels, one may be appreciably misled.

For the dual problem of estimating component reliability based on data from a series system, it appears that departures from independence are of a greater consequence. Both parametric and nonparametric estimators of relevant component parameters are inconsistent. Although under independence, the bias of the estimators of interest clouds the issues, it is clear that for larger negative correlations these estimators tend to underestimate the parameter, whereas for large positive correlations, the reverse is true.

ACKNOWLEDGMENTS

The authors would like to thank the editor and the referees for their constructive comments.

This work was supported by the Air Force Office of Scientific Research under Contract AFOSR-82-0307 and the Graduate School of the Ohio State University.

[*Received June 1982. Revised December 1983.*]

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Appendix B

A TEST FOR INDEPENDENCE BASED ON DATA

FROM A BIVARIATE SERIES SYSTEM

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30 September, 1984

Annual Report for Period 1 October 1983 - 30 September, 1984

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A TEST FOR INDEPENDENCE BASED ON DATA FROM A BIVARIATE SERIES SYSTEM

by

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AMS 1980 Subject Classifications. Primary 62G10, Secondary 62N05.

Key Words and Phrases: Test for Independence, Series Systems, Identifiability.

Technical Report No. 285
Department of Statistics
The Ohio State University
Columbus, Ohio 43210

June 1984

Abstract

The problem of testing for independence of the component lifetimes when the components are linked in series is considered. To avoid the problem of nonidentifiability the marginal component lifetimes are assumed to be known. In this setting a modified version of Kendall's Tau is proposed. This test statistic is obtained by replacing those component lifetimes which cannot be observed, due to system failure, by conditional probabilities computed under independence. A small scale simulation study of the power of this test shows the test has reasonable power for relatively small sample sizes.

Key Words: Series Systems; Test for Independence; Kendall's Tau; Exponential Distribution.

1. DEPENDENT SYSTEMS

A common assumption made in modeling series systems is that the component lifetimes are statistically independent. This assumption is also routinely made in analyzing data collected from series systems. Recently, Klein and Moeschberger (1983) and Moeschberger and Klein (1984) have shown that one may be appreciably misled by this independence assumption for certain bivariate exponential systems.

To illustrate the effects of this independence assumption consider the following two models for the joint survival function of the component lifetimes (X,Y). The first model, due to Oakes (1982) has joint survival function

$$H(x,y) = P(X>x,\ Y>y) = \left[\left\{\frac{1}{\overline{F}(x)}\right\}^{\theta-1} + \left\{\frac{1}{\overline{G}(y)}\right\}^{\theta-1} - 1\right]^{-1/(\theta-1)},\ \theta > 1 \qquad (1.1)$$

where $\overline{F}(\cdot)$, $\overline{G}(\cdot)$ are the marginal survival functions of X and Y respectively. This distribution has a coefficient of concordance $\tau = (\theta-1)/(\theta+1)$ and $\theta = 1$ corresponds to independent component failure times. If $\lambda(x|Y=y)$ and $\lambda(x|Y>y)$ denote the conditional hazard functions for the conditional distributions of X given Y = y and given Y > y, respectively then $\lambda(x|Y=y) = \theta\ \lambda(x|Y>y)$.

A second model, due to Gumbel (1960), has joint survival function

$$\overline{H}(x,y) = \overline{F}(x)\overline{G}(y)[1+\alpha(1-\overline{F}(x))(1-\overline{G}(y))],\ -1 < \alpha < 1. \qquad (1.2)$$

This model has coefficient of concordance $\tau = 2\alpha/9$ which, unlike the Oakes model, may be both positive and negative.

To illustrate the importance of the independence assumption in modeling the system life consider figures 1 and 2 where the 95th and 99th

percentile of system life is plotted for the two models with exponential marginals. Here in all cases the first component has unit mean life. For the Gumbel model the true percentile ranges from 80% to 115% of the percentile computed under independence, while in the Oakes model the true percentile can be as much as twice as big as the percentile computed under independence when $\lambda_2 = \lambda_1$ and as much as 1.5 time as big when $\lambda_2 = 2$.

Since one may be appreciably misled by erroneously assuming independent component lifetimes it is desirable to test this hypothesis based on data from series systems. However, if no assumptions about the underlying distribution of the component lifetimes is made such a test is impossible due to the identifiability problem (see, e.g. Tsiatis (1978), Miller (1977), Basu and Klein (1982)). This is, given any set of observable information (such as system life, crude system life, etc.) collected from a series system with dependent component lifetimes, there exists a series system with independent component lifetimes with the same observable information (see Langberg, Proschan and Quinzi (1981)). However, this comparable system of independent random variables need not have the same marginal component life distributions as the dependent structure. In particular, the marginal distributions of the two systems are the same only for the class of constant sum models defined by Williams and Lagakos (1977).

In the next section a modification of Kendall's (1938) test for independence is proposed. This test assumes that the marginal component life distributions are completely specified. This information could be obtained by testing each component seperately, as is often done in the development stages of system design (see, e.g. Easterling and Prairie (1971), Mastran (1976), or Miyamura (1982)). In section 3 a simulation study compares the power of this test to the parametric tests based on Oakes and Gumbel models.

2. THE TEST PROCEDURE

Suppose that n two component series systems are put on test. Let X_i, Y_i denote the potential (unobservable) failure times of the first and second components of the i^{th} system. We are not allowed to observe (X_i, Y_i) directly, but instead we observe $T_i = \min(X_i, Y_i)$, the system failure time and

$$I_i = \begin{cases} 1 & \text{if } T_i = X_i \\ 0 & \text{if } T_i = Y_i \end{cases}$$, the cause of the system failure.

Also suppose that the marginal survival functions of X_i and Y_i, $\overline{F}(x) = P(X_i>x)$ and $\overline{G}(y) = P(Y_i>y)$, $i = 1,\ldots,n$ are known.

If we could observe both X_i and Y_i then a test of independence, due to Kendall (1938), is to count the number of concordant pairs and the number of discordant pairs. A pair (X_i, Y_i), (X_j, Y_j) is concordant if $X_i - X_j$ and $Y_i - Y_j$ have the same sign and is discordant if these differences have different signs. The test statistic is then the number of concordant pairs minus the number of discordant pairs.

If the data comes from a series system then only T_i, I_i is observed. Suppose we consider a pair (T_i, I_i), (T_j, I_j) with $T_i < T_j$. If $I_i = 1$ and $I_j = 1$ then we know that $X_i = T_i < X_j = T_j$, and $X_i < Y_i$, $X_j < Y_j$. This pair would be concordant, regardless of the value of Y_j, if $T_i < Y_i < T_j$. If $Y_i > T_j$ concordance or discordance depends on the value of Y_j. Under the null hypothesis of independence, the conditional probability that the pair is concordant is $[\overline{G}(T_i) - \overline{G}(T_j)]/\overline{G}(T_i) = P(T_i < Y < T_j \mid Y > T_i)$ since average concordance over the range $Y > T_j$ is 0. When $I_i = 1$ and $I_j = 0$ then $T_i = X_i < Y_j = T_j$, $X_i < Y_i$, $Y_j < X_j$. Here if $T_i < Y_i < T_j$ the pair would be concordant and if $Y_i > T_j$ the pair would be discordant, whatever the value of X_j. Under independence the conditional probabilities of these two events are $[\bar{G}(T_i) - \bar{G}(T_j)]/\bar{G}(T_i)$ and $\bar{G}(T_j)/\bar{G}(T_i)$, respectively. Should $I_i = 0$ similar probabilities, involving $\bar{F}$, could be obtained. This motivation suggests the following score function for $T_i < T_j$

$$\phi(T_i,I_i,T_j,I_j) = \begin{cases} [\bar{G}(T_i)-\bar{G}(T_j)]/\bar{G}(T_i) & \text{if } I_i = I_j = 1 \\ [\bar{F}(T_i)-\bar{F}(T_j)]/\bar{F}(T_i) & \text{if } I_i = I_j = 0 \\ [\bar{G}(T_i)-2\bar{G}(T_j)]/\bar{G}(T_i) & \text{if } I_i = 1,\ I_j = 0 \\ [\bar{F}(T_i)-2\bar{F}(T_j)]/\bar{F}(T_i) & \text{if } I_i = 0,\ I_j = 1 \end{cases} \tag{2.1}$$

and similarly for $T_i > T_j$.

The modified version of Kendall's test statistic is

$$\hat{\tau} = \sum_{1\leq i<j\leq n}\sum \phi(T_i,I_i,T_j,I_j)/\binom{n}{2}. \tag{2.2}$$

To find the moments of $\hat{\tau}$, under independence, consider the pairs (T_1, I_1), (T_2, I_2). Let $A_1 = \{T_1 < T_2, I_1 = I_2 = 1\}$, $A_2 = \{T_1 < T_2, I_1 = 1, I_2 = 0\}$, $A_3 = \{T_1 < T_2, I_1 = 0, I_2 = 0\}$ and $A_4 = \{T_1 < T_2, I_1 = 0, I_2 = 1\}$. In terms of the unobservable component lifetimes, (X_i, Y_i), $A_1 = \{X_1 < X_2, X_1 < Y_1, X_2 < Y_2\}$, $A_2 = \{X_1 < Y_2, X_1 < Y_1, Y_2 < X_2\}$, $A_3 = \{Y_1 < Y_2, Y_1 < X_1, Y_2 < X_2\}$, and $A_4 = \{Y_1 < X_2, Y_1 < X_1, X_2 < Y_2\}$. Since, by symmetry T_1 is equally likely to be either smaller or larger than T_2 we have

$$\frac{1}{2} E(\phi(T_1,I_1,T_2,I_2)) = \int_{A_1} \frac{\bar{G}(x_1)-\bar{G}(x_2)}{\bar{G}(x_1)} dF(x_1)dF(x_2)dG(y_1)dG(y_2)$$

$$+ \int_{A_2} \frac{\bar{G}(x_1)-2\bar{G}(y_2)}{\bar{G}(x_1)} dF(x_1)dF(x_2)dG(y_1)dG(y_2)$$

$$+ \int_{A_3} \frac{\bar{F}(y_1)-\bar{F}(y_2)}{\bar{F}(y_1)} dF(x_1)dF(x_2)dG(y_1)dG(y_2) \tag{2.3}$$

$$+ \int_{A_4} \frac{\bar{F}(y_1)-2\bar{F}(x_2)}{\bar{F}(y_1)} dF(x_1)dF(x_2)dG(y_1)dG(y_2).$$

$$= J_1 + J_2 + J_3 + J_4 \quad \text{(say)}.$$

Now, consider

$$J_1 + J_2 = \int_{-\infty}^{\infty} \{\int_x^{\infty}[\bar{G}(x)-\bar{G}(y)]\bar{G}(y)dF(y) + \int_x^{\infty}[\bar{G}(x)-2\bar{G}(y)]\bar{F}(y)dG(y)\}dF(x). \tag{2.4}$$

Integrating the first inner integral in (2.4) by parts yields the negative of the second inter integral so that $J_1 + J_2 = 0$. Similar computations show that $J_3 + J_4 = 0$. Thus $E(\phi(T_1, I_1, T_2, I_2))$ and hence $E(\hat{\tau})$ are both 0. By similar computations one can show that

$$n(n-1)V(\hat{\tau}) = \frac{4}{3}\int_{-\infty}^{\infty} \bar{G}(x)^2 dF(x) + \frac{4}{3}\int_{-\infty}^{\infty} \bar{F}(x)^2 dG(x)$$

$$- 4\int_{-\infty}^{\infty} \bar{G}(x)^{-1}\int_x^{\infty} F(y)\bar{G}(y)^2 dG(y)dF(x)$$

$$- 4\int_{-\infty}^{\infty} \bar{F}(x)^{-1}\int_x^{\infty} G(y)\bar{F}(y)^2 dF(y)dG(x)$$

$$+ 4(n-2)\{\frac{2}{3} - 2\int_{-\infty}^{\infty} F(x)G(x)d(F(x) + G(x)) \tag{2.5}$$

$$- 2\int_{-\infty}^{\infty} F(x)^2 G(x)^2 d(F(x) + G(x))$$

$$+ 3\int_{-\infty}^{\infty} F(x)^2 G(x)dF(x) + 3\int_{-\infty}^{\infty} F(x)G(x)^2 dG(x)\},$$ where $F(x) = 1 - \bar{F}(x)$ and $G(x) = 1 - \bar{G}(x)$.

The asymptotic normality of $\hat{\tau}$ follows by the results of Hoeffding (1948). Hence, a test of independence versus dependence rejects if $|\hat{\tau}/\sqrt{V(\hat{\tau})}|$ is greater than the appropriate percentage point of a standard normal random variable. A test of independence versus positive dependence rejects if $\hat{\tau}/\sqrt{V(\hat{\tau})}$ is too large.

The variance of $\hat{\tau}$ (2.5) can be expressed explicitly in several cases.

Case 1. $\bar{F}(x) = \bar{G}(x)$. In this case (2.5) reduces to

$$V(\hat{\tau}) = \frac{4n+7}{30n(n-1)} \,. \tag{2.6}$$

Case 2. (Lehmann structure) $\bar{F}(x) = \bar{G}(x)^{\alpha}$. Here (2.5) reduces to

$$n(n-1)V(\hat{\tau}) = \frac{8\alpha[35\alpha + n(9\alpha^2+2\alpha+9)]}{3(3\alpha+1)(3+\alpha)(2\alpha+3)(3\alpha+2)} \,. \tag{2.7}$$

Case 3. (X, Y exponential), $\bar{F}(X) = e^{-\lambda x}$, $\bar{G}(y) = e^{-\theta y}$, then (2.5) reduces to

$$n(n-1)V(\hat{\tau}) = \frac{8\lambda\theta[35\lambda\theta + n(9\lambda^2+2\lambda\theta+9\theta^2)]}{3(3\lambda+\theta)(\lambda+3\theta)(2\lambda+3\theta)(3\lambda+2\theta)} \,. \tag{2.8}$$

When the true values of $\bar{F}$, $\bar{G}$ are misspecified then $E(\hat{\tau})$ is not zero. If the true component lifetime distributions are $\bar{F}$, $\bar{G}$ but $\bar{F}^{\alpha}$, $\bar{G}^{\beta}$ are used in formula (2.3) then one can show that, under independence,

$$\begin{aligned} E(\hat{\tau}) &= 2(1-\beta)\int_{-\infty}^{\infty} \bar{G}(x)^{1-\beta} \int_{x}^{\infty} \bar{F}(y)\bar{G}(y)^{\beta}\,dG(y)\,dF(x) \\ &\quad + 2(1-\alpha)\int_{-\infty}^{\infty} \bar{F}(x)^{1-\alpha} \int_{x}^{\infty} \bar{G}(y)\bar{F}(y)^{\alpha}\,dF(y)\,dG(x), \quad \alpha, \beta > 0. \end{aligned} \tag{2.9}$$

If $\bar{F}(\cdot) = \bar{G}(\cdot)$ then $E(\hat{\tau}) = \dfrac{\beta-1}{2(\beta+2)} + \dfrac{\alpha-1}{2(\alpha+2)}$.

If $\bar{F}(x) = \bar{G}(x)^{\theta}$ then $E(\hat{\tau}) = \dfrac{\theta}{(\theta+1)}\left\{\dfrac{\alpha-1}{\theta+\alpha\theta+1} + \dfrac{\beta-1}{\theta+\beta+1}\right\}$

Similar expressions can be obtained for the null variance of $\hat{\tau}$.

3. SIMULATION STUDY

To study the effectiveness of the modified Kendall's τ described in section 2 a simulation study was conducted. The study was performed by generating 1000 samples of $n = 20$ or 40 series systems with exponentially distributed component life times, $\bar{F}(x) = e^{-x}$ and $\bar{G}(y) = e^{-\lambda_2 y}$, $\lambda_2 = 1, 2$. Both the Oakes joint distribution (1.1) and the bivariate Gumbel distribution (1.2) were used. The bivariate observations from the Oakes distribution were generated using the technique described in section 2 of that paper. To generate Gumbel random variables with marginal survival functions $\bar{F}(x)$, $\bar{G}(y)$ let U_1, U_2 be independent uniform random deviates. Note that

$$\bar{F}(x|y) = P(X>x|Y=y) = \bar{F}(x)(1+\alpha-2\alpha\bar{G}(y)) - \alpha\bar{F}(x)^2(1-2\bar{G}(y)). \qquad (3.1)$$

Let $U_1 = \bar{G}(y)$ and $U_2 = \bar{F}(x|y) = \bar{F}(x)[1+\alpha-2\alpha U_1] - \alpha\bar{F}(x)^2(1-2U_1)$.

Solving this equation for $\bar{F}(x)$ yields

$$\bar{F}(x) = U^* = \frac{(1+\alpha(1-2U_1)) - \alpha(1+\alpha^2(1-2U_1)^2 + 2\alpha(1-2U_1)(1-2U_2))^{1/2}}{2\alpha(1-2U_1)}, \quad U_1 \neq 1/2 \qquad (3.2)$$

which is the root which lies in the interval $[0,1]$. If $U_1 = 1/2$ then $U^* = U_2$. The pair $(\underline{X}, \underline{Y})$ is then found by $X = \bar{F}^{-1}(U^*)$, $Y = \bar{G}^{-1}(U_1)$.

For the purpose of comparison the parametric tests for independence, based on the efficient scores statistics, for the Gumbel and Oakes model were obtained. Consider first the Gumbel model (1.2). Using the notation in section 2, the observable crude density for $I = 1$ is

$$\frac{-d}{dt} P(T>t, I=1) = q_1(t) = f(t)\bar{G}(t)[1+\alpha(1-\bar{F}(t)-2\bar{G}(t) + \bar{F}(t)\bar{G}(t))]$$

where $f(t) = \frac{-d}{dt}\bar{F}(t)$, and a similar expression for $q_0(t)$ when $I = 0$. Based on a sample of n series systems the likelihood function is

$$L(\alpha) \propto \prod_{j=1}^{n} q_1(t_j)^{I_j} q_0(t_j)^{1-I_j}. \qquad (3.3)$$

$$Z = \left(\sum_{j=1}^{n} I_j(\lambda_1 T_j) + (1-I_j)\lambda_2 T_j + \sum_{j=1}^{n} \lambda_1 \lambda_2 T_j^2 - (\lambda_1+\lambda_2)T_j) \right) \cdot \frac{\lambda_1 + \lambda_2}{(2n\lambda_1\lambda_2)^{1/2}}$$

which is approximately standard normal for large n when $\theta = 1$.

The results of this study are reported in table 1. From this table it seems like the modified τ test has reasonably good power when compared to the parametric tests, although comparison with the Oakes score test is hard since the significance level of that test is inflated. Also the test based on the Gumbel scores has comparable power when the data is from the Oakes model. A test for normality done on the samples where the components were independent accepted the normality assumption for the modified τ test.

Table 2 reports the observed number of rejections when the component parameters are estimated based on independent samples of size 50 for each component. A .05 significance level was used. Here, when $\lambda_1 = \lambda_2$, all tests have inflated levels. When $\lambda_1 \neq \lambda_2$ the tests are conservative. All tests have comparable power when $\lambda_1 = \lambda_2$, however the modified τ test has significantly higher power when $\lambda_1 \neq \lambda_2$.

In addition to the power of our modified test the $E(\hat{\tau})$ was estimated for each sample. Except in the independence case the simulation showed that $E(\hat{\tau}) \cong .35\tau$, suggesting $\hat{\tau}$ is of limited use as a point estimator of τ.

ACKNOWLEDGEMENTS

The author would like to thank Professor M. L. Moeschberger of The Ohio State University and P. Laud of Northern Illinois University for their constructive comments. This research was supported by contract AFOSR-82-0307 for the Air Force Office of Scientific Research.

TABLE 1 Estimated Power Using True Parameter Values Based On 1000 Replications

MODEL	n	τ	MODIFIED τ α=.05	MODIFIED τ α=.025	OAKES SCORE α=.05	OAKES SCORE α=.025	GUMBEL SCORE α=.05	GUMBEL SCORE α=.025
Independent (λ_2=1)	20	0	50	25	71+	52+	58	37
Independent (λ_2=1)	40	0	42	21	57	34	54	36
Independent (λ_2=2)	20	0	53	25	74+	52+	62	34
Independent (λ_2=2)	40	0	55	32	74+	49+	69	37
Gumbel (λ_2=1)	20	.125	99	55	115	88	133	78
Gumbel (λ_2=1)	40	.125	158	96	159	124	192	124
Gumbel (λ_2=2)	20	.125	119	74	128	88	141	100
Gumbel (λ_2=2)	40	.125	158	88	146	111	176	116
Gumbel (λ_2=1)	20	.222	182	117	172	130	245	175
Gumbel (λ_2=1)	40	.222	283	199	239	179	323	257
Gumbel (λ_2=2)	20	.222	188	110	160	130	205	143
Gumbel (λ_2=2)	40	.222	273	181	221	159	316	237
Oakes (λ_2=1)	20	.125	170	114	236	202	184	137
Oakes (λ_2=1)	40	.125	224	154	327	273	247	185
Oakes (λ_2=2)	20	.125	166	101	231	207	179	125
Oakes (λ_2=2)	40	.125	228	148	313	253	248	166
Oakes (λ_2=1)	20	.25	318	243	421	377	379	295
Oakes (λ_2=1)	40	.25	484	394	614	551	510	443
Oakes (λ_2=2)	20	.25	334	223	386	335	354	273
Oakes (λ_2=2)	40	.25	513	338	555	483	522	407
Oakes (λ_2=1)	20	.50	638	535	704	670	680	606
Oakes (λ_2=1)	40	.50	880	802	903	875	881	851
Oakes (λ_2=2)	20	.50	657	589	615	547	674	593
Oakes (λ_2=2)	40	.50	894	823	816	772	873	820
Oakes (λ_2=1)	20	.75	799	722	803	763	858	795
Oakes (λ_2=1)	40	.75	973	946	983	968	925	900
Oakes (λ_2=2)	20	.75	899	847	699	631	823	763
Oakes (λ_2=2)	40	.75	995	989	924	884	985	961

Figure 1

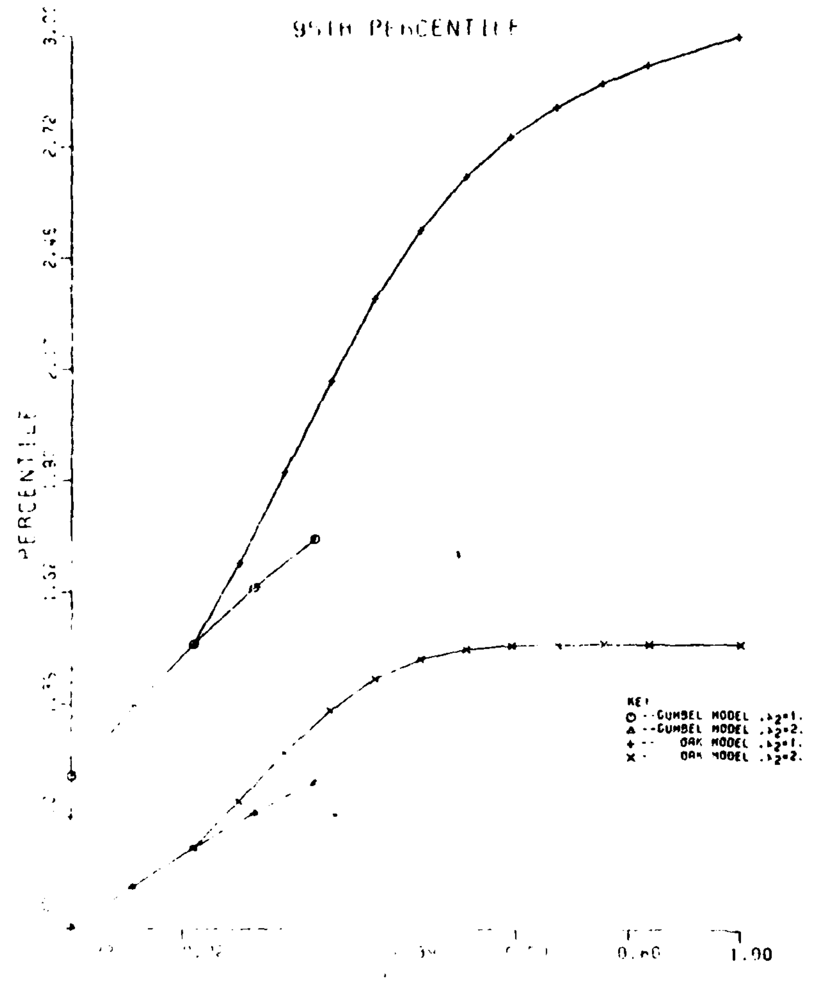

Figure 2

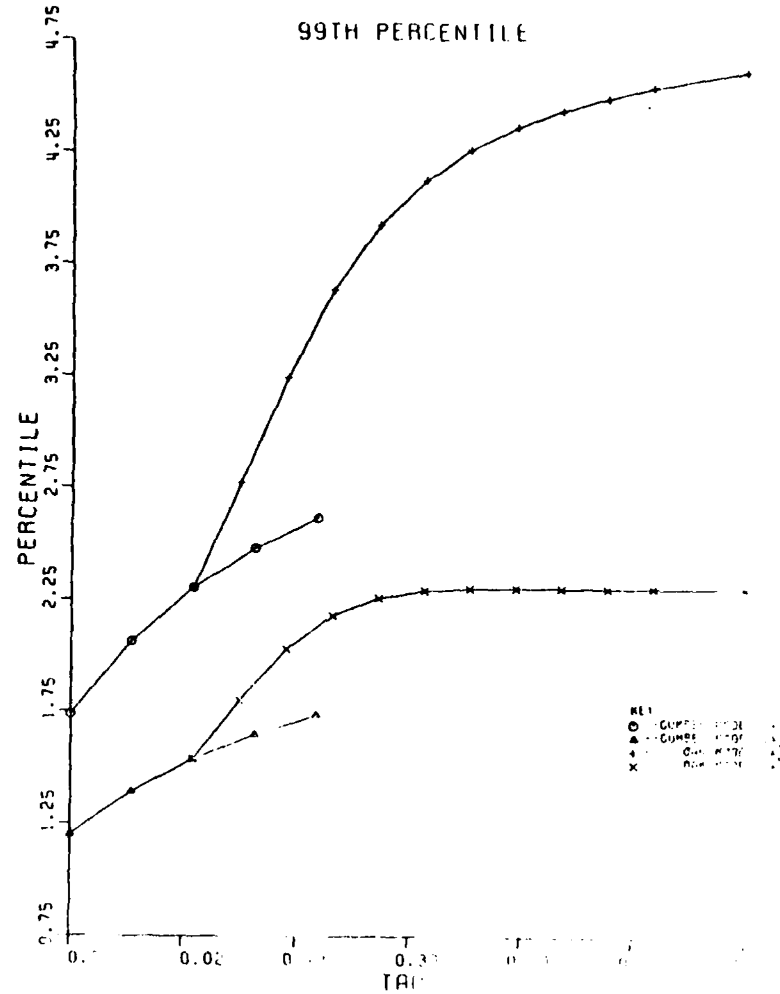

TABLE : Estimated Power Using True Parameter Values
Based On 1000 Replications

MODEL	n	τ	MODIFIED τ $\alpha=.05$	MODIFIED τ $\alpha=.025$	OAKES SCORE $\alpha=.05$	OAKES SCORE $\alpha=.025$	GUMBEL SCORE $\alpha=.05$	GUMBEL SCORE $\alpha=.025$
Independent ($\lambda_2=1$)	20	0	50	25	71+	52+	58	37
Independent ($\lambda_2=1$)	40	0	42	21	57	34	54	36
Independent ($\lambda_2=2$)	20	0	53	25	74+	52+	62	34
Independent ($\lambda_2=2$)	40	0	55	32	74+	49+	69	37
Gumbel ($\lambda_2=1$)	20	.125	99	55	115	88	133	78
Gumbel ($\lambda_2=1$)	40	.125	158	96	159	124	192	124
Gumbel ($\lambda_2=2$)	20	.125	119	74	128	88	141	100
Gumbel ($\lambda_2=2$)	40	.125	158	88	146	111	176	116
Gumbel ($\lambda_2=1$)	20	.222	182	117	172	130	245	175
Gumbel ($\lambda_2=1$)	40	.222	283	199	239	179	323	257
Gumbel ($\lambda_2=2$)	20	.222	188	110	160	130	205	143
Gumbel ($\lambda_2=2$)	40	.222	278	181	221	159	316	237
Oakes ($\lambda_2=1$)	20	.125	170	114	236	202	184	137
Oakes ($\lambda_2=1$)	40	.125	224	154	327	273	247	185
Oakes ($\lambda_2=2$)	20	.125	166	101	231	207	179	125
Oakes ($\lambda_2=2$)	40	.125	228	148	313	253	248	166
Oakes ($\lambda_2=1$)	20	.25	318	243	421	377	379	295
Oakes ($\lambda_2=1$)	40	.25	484	394	614	551	510	443
Oakes ($\lambda_2=2$)	20	.25	334	223	386	335	354	273
Oakes ($\lambda_2=2$)	40	.25	513	338	555	483	522	407
Oakes ($\lambda_2=1$)	20	.50	638	535	704	670	680	606
Oakes ($\lambda_2=1$)	40	.50	880	802	903	875	881	851
Oakes ($\lambda_2=2$)	20	.50	657	589	615	547	674	593
Oakes ($\lambda_2=2$)	40	.50	894	823	816	772	873	820
Oakes ($\lambda_2=1$)	20	.75	799	722	803	763	858	795
Oakes ($\lambda_2=1$)	40	.75	973	946	983	968	925	900
Oakes ($\lambda_2=2$)	20	.75	899	847	699	631	823	763
Oakes ($\lambda_2=2$)	40	.75	995	989	924	884	985	961

TABLE 2 Estimated Power Using Estimated Parameter Values and 0.05 Significance Level

MODEL	n	τ	Modified τ $\lambda_1=1$	Modified τ $\lambda_2=2$	Oakes Score $\lambda_1=1$	Oakes Score $\lambda_2=2$	Gumbel Score $\lambda_1=1$	Gumbel Score $\lambda_2=2$
Independent	20	0	83	18	96	1	89	1
Independent	40	0	64	17	83	1	81	1
Gumbel	20	.125	136	42	141	3	159	4
Gumbel	40	.125	214	30	200	3	242	4
Gumbel	20	.22	209	68	201	6	255	10
Gumbel	40	.22	331	62	276	10	360	9
Oakes	20	.125	202	48	263	40	211	8
Oakes	40	.125	276	54	354	13	284	1
Oakes	20	.25	327	146	430	55	388	29
Oakes	40	.25	513	156	628	48	542	20
Oakes	20	.50	638	400	699	93	655	92
Oakes	40	.50	858	558	827	99	865	102
Oakes	20	.75	781	737	793	76	828	84
Oakes	40	.75	956	916	947	112	962	138

Appendix C

BOUNDS ON NET SURVIVAL PROBABILITIES
FOR DEPENDENT COMPETING RISKS

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SUMMARY

Improved bounds on the marginal survival function based on data from a competing risk experiment are obtained. These bounds are obtained by specifying a range of possible concordances for the risks. These bounds are tighter than those of Peterson (1976). A comparison to other existing bounds is also made.

Key Words: Competing Risks, Product Limit Estimator, Net Survival Function, Coefficient of Concordance.

Author's Footnote

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I. INTRODUCTION

A common problem in survival analysis is to estimate the marginal survival function of the time, X, until some event such as remission, component failure, or death due to a specific cause occurs. Often observation of this main event of interest is impossible due to the occurrence of a competing risk at some time $Y < X$, such as censoring, failure of a different component in a series system, or death from some cause not related to the study. Standard statistical methods, which assume these competing risks are independent, estimate the marginal survival function by the Product Limit Estimator of Kaplan and Meier (1958). This estimator has been shown to be consistent for the marginal survival function by Langberg, Proschan and Quinzi (1981) when the risks follow a constant sum model defined by Williams and Lagakos (1977). When the risks are not in the class of constant sum models, the Product Limit Estimator is inconsistent and, in such cases, the investigator may be appreciably misled by assuming independence.

In the competing risks framework we observe $T = \text{minimum}(X,Y)$ and $I = \chi(X \leq Y)$ where $\chi(\cdot)$ denotes the indicator function. Tsiatis (1975) and others have shown that the pair (T,I) provides insufficient information to determine the joint distribution of X and Y. That is, there exists both an independent and a dependent model for (X,Y) which produces the same joint distribution for (T,I). However, these "equivalent" independent and dependent joint distributions may have quite different marginal distributions. Also, due to this identifiability problem, there may be several dependent models with different marginal structures which will yield the same observable information, (T,I). In light of the consequences of the untestable independence assumption in using the Product Limit estimator to estimate the marginal survival function of X, it is important to have bounds on this function based on the observable random variables (T,I) and some assumptions on the joint behavior of X and Y.

Peterson (1976) has obtained general bounds on the marginal survival function of X, $S(x)$, based on the estimable joint distribution of (T,I). Let $Q_1(x) = P(T > x, I = 1)$, and $Q_2(x) = P(T > x, I = 0)$ be the crude survival functions of T. His bound, obtained from the limits on the joint distribution of (X, Y) obtained by Fréchet (1951), is

$$Q_1(x) + Q_2(x) \leq S(x) \leq Q_1(x) + Q_1(0). \tag{1.1}$$

Since these bounds allow for any dependence structure, they can be very wide and provide little useful information to an investigator.

Fisher and Kanarek (1974) have obtained tighter bounds on $S(x)$ in terms of a dependence measure α. Their model assumes that simultaneous to the occurrence of Y an event occurs which either stretches or contracts the remaining life of X by an amount associated with α. That is, $P(X > x|Y = y < x) = P(X > y + \alpha(x-y)|Y > y + \alpha(x-y))$. A large α, for example, implies that a small survival after censoring is the same as α-times as much survival if censoring was not present. They show that if α is assumed known, then the marginal survival function can be estimated from the observable information. Also these estimates, $\hat{S}_{\alpha}(x)$, are decreasing in α. For their bounds, the investigator specifies a range of possible values $\alpha_L < \alpha < \alpha_u$ so that $S_{\alpha_u}(x) \leq S(x) \leq S_{\alpha_L}(x)$.

Recently, Slud and Rubenstein (1983), have proposed general bounds. They show that knowledge of the function

$$\rho(x) = \lim_{\delta \to 0} \frac{P(x < X < x + \delta | X > x, Y \leq x)}{P(x < X < x + \delta | X > x, Y > x)}$$

along with the observable information (T,I) is sufficient to uniquely determine the marginal distribution of X. These estimates $\hat{S}_{\rho}(t)$ are decreasing functions of ρ for fixed x. Their bounds are obtained by specifying a range of possible values $\rho_1(x) \leq \rho(x) \leq \rho_2(x)$ so that if $\rho(x)$ is the true function $\hat{S}_{\rho_2}(x) \leq S(x) \leq \hat{S}_{\rho_1}(x)$.

In this paper we obtain different bounds on the marginal survival function by assuming a particular dependence structure on X and Y. These bounds are functions of the observables (T,I) and a familiar dependence measure, the concordance probability between X and Y. In Section 2 we describe this model in detail. In Section 3 we derive the bounds and show consistency when the dependence parameter is known. In section 4 these bounds are compared to those obtained by Peterson, Fisher and Kanarek, and Slud and Rubenstein.

II. THE MODEL

The dependence structure we shall employ to model the joint survival was first introduced by Clayton (1978) to model association in bivariate lifetables and, later, by Oakes (1982) to model bivariate survival data. Let $S(x) = P(X \geq x)$, $R(y) = P(Y \geq y)$, with $S(0) = R(0) = 1$, be the continuous univariate survival functions of the death and censoring times, respectively. For $\theta \geq 1$ define $F(x,y) = P(X > x, Y > y)$ by

$$F(x,y) = [\{\frac{1}{S(x)}\}^{\theta-1} + \{\frac{1}{R(y)}\}^{\theta-1} - 1]^{-1/(\theta-1)} \tag{2.1}$$

This joint distribution has marginals S and R. As $\theta \to 1$, then (2.1) reduces to the joint distribution with independent marginals. For $\theta \to \infty$, $F(x,y) \to \min(S(x), R(y))$ the bivariate distribution with maximal positive association for these marginals. The probability of concordance is $\theta/(\theta + 1)$ so that Kendall's (1962) coefficient of concordance is $\tau = (\theta - 1)/(\theta + 1)$ which spans the range 0 to 1.

This model has a nice physical interpretation in terms of the functions $\lambda(x|Y = y)$ and $\lambda(x|Y > y)$, the hazard functions of X given $Y = y$ and X given $Y > y$, respectively. From (2.1) one can show that

$$\lambda(x|Y = y) = \theta\lambda(x|Y > y)$$

or

$$P(X > x|Y = y) = [P(X > x|Y > y)]^{\theta}. \tag{2.2}$$

For $\theta > 1$ the hazard rate of survival if censoring occurs at time y is θ times the hazard rate of survival if censoring does not occur at time y. This implies that the hazard rate after censoring occurs is accelerated by a factor of θ over the hazard rate if censoring had not occurred. Also when $\theta = 1$, (2.2), reduces to the condition required by Williams and Lagakos (1977) for a model to be constant sum and hence for the usual product limit estimator of S(t) to be consistent (See Basu and Klein (1982) for details).

Oakes (1982) also shows that (2.1) can be obtained from the following random effects model. Let $S^*(x) = \exp\{-[\frac{1}{S(x)}]^{\theta-1} + 1\}$ and let $R^*(y)$ be similarly defined. Let W have a gamma distribution with density $g(w) \propto w^{\frac{1}{\theta-1}-1} e^{-w}$ and conditional on W = w let X,Y be independent with survival functions $\{S^*(x)\}^w$ and $\{R^*(y)\}^w$. Then, unconditionally, X,Y have the joint survival function F(x,y) given by (2.1).

For fixed marginals S and R the joint probability density function, f(x,y), can be shown to be totally positive of order 2 for all $\theta \geq 1$. This implies that (X,Y) are positive quadrant dependent. In particular, one can show that for S,R fixed the family of distributions $\mathcal{F} = \{F(x,y): \theta \geq 1\}$ is increasing positive quadrant dependent in θ as defined by Ahmed, et al. (1979).

III. BOUNDS ON MARGINAL SURVIVAL

Suppose that X and Y have the joint distribution (2.1) and let $T = \min(X,Y)$, then the survival function of T is

$$F(T) = [[\frac{1}{S(t)}]^{\theta-1} + [\frac{1}{R(t)}]^{\theta-1} - 1]^{-\frac{1}{\theta-1}} \tag{3.1}$$

and the crude density function associated with X,

$q_1(t) = \frac{d}{dt} P(T < t, X < Y)$, is given by

$$q_1(t) = \frac{s(t)}{S^{\theta}(t)} [F(t)]^{\theta}, \tag{3.2}$$

where $s(t) = -dS(t)/dt$.

Now consider the differential equation

$$s(t)/S^{\theta}(t) = q_1(t)/[F(t)]^{\theta} \tag{3.3}$$

and suppose θ is known. Then the solution of (3.3) for S(t) is

$$\begin{aligned} S_{\theta}(t) &= \left[1 + (\theta-1) \int_0^t \frac{q_1(u)}{[F(u)]^{\theta}} du\right]^{-(\frac{1}{\theta-1})} \quad \text{if } \theta > 1 \\ &= \exp\left(- \int_0^t \frac{q_1(u)}{F(u)} du\right) \quad \text{if } \theta = 1. \end{aligned} \tag{3.4}$$

The functions $F(\cdot)$ and $q_1(\cdot)$ are directly estimable from the data one sees in a competing risks experiment. Let $T_1, \ldots, T_n$ denote the observed test times of n individuals put on test and let I_i, $i = 1, \ldots, n$ be 1 or 0 according to whether the T_i was an observation on X_i or Y_i, respectively.

Define $\hat{F}(t) = \sum_{i=1}^{n} \chi(T_i > t)/n$ and $\hat{Q}_1(t) = \sum_{i=1}^{n} \chi(T_i \le t,\ I_i = 1)$.

Then if θ is known, a natural estimator of $S_\theta(t)$ is

$$\hat{S}_\theta(t) = [1 + (\theta-1) \int_0^t \frac{d\hat{Q}_1(u)}{[\hat{F}(u)]^\theta}]^{-\frac{1}{(\theta-1)}} \quad \text{if } \theta > 1$$

$$= \exp(-\int_0^t \frac{d\hat{Q}_1(u)}{\hat{F}(u)}) \quad \text{if } \theta = 1 \tag{3.5}$$

For $\theta = 1$, this estimator is of the form of the hazard rate estimator proposed by Nelson (1972). The estimators (3.5) can be expressed in the following form for computation purposes.

$$\hat{S}_\theta(t) = \exp\left\{-\sum_{T_{(i)} \le t_i,\, I_{(i)} = 1} \frac{1}{(n-i+1)}\right\} \quad \text{if } \theta = 1$$

$$[1 + (\theta-1)n^{\theta-1} \sum_{T_{(i)} < t,\, I_{(i)} = 1} \frac{1}{(n-i+1)} \theta]^{-\frac{1}{\theta-1}} \tag{3.6}$$

$$\text{if } \theta > 1$$

where $T_{(1)}, \ldots, T_{(n)}$ are the ordered death times.

For θ known and if the true underlying joint distribution of (X,Y) is of the form (2.1) then $\hat{S}_\theta(t)$ is a consistent estimator of $S(t)$ as shown by the following theorem.

<u>Theorem 1</u>. Let (X,Y) have the form (2.1) with marginals $S(t)$, $R(t)$ respectively. Let $\theta \ge 1$ be known. Then on the set where $S(t) > 0$ we have $\hat{S}_\theta(t) \to S(t)$ a.s.

Proof:

For $\theta = 1$, the result follows by a theorem of Langberg, Proschan and Quinzi (1981). Suppose that $\theta > 1$. Note that $\hat{Q}_1(t) \to Q_1(t)$ a.s. and $\hat{F}(u) \to F(u)$ a.s. by the strong law of large numbers. Since $\hat{S}_\theta(t)$ is a continuous function of $\int_0^t \frac{d\hat{Q}_1(u)}{[\hat{F}(u)]^\theta}$ in the support of $\hat{F}(u)$, it suffices to show

$$\int_0^t \frac{d\hat{Q}_1(u)}{[\hat{F}(u)]^\theta} \to \int_0^t \frac{dQ_1(u)}{[F(u)]^\theta} \quad \text{a.s.}$$

Now, after an integration by parts,

$$\int_0^t \frac{d\hat{Q}_1(u)}{[\hat{F}(u)]^\theta} = \frac{\hat{Q}_1(t)}{[\hat{F}(t)]^\theta} - \int_0^t \hat{Q}_1(u)\, d\left(\frac{1}{\hat{F}^\theta(u)}\right)$$

$$= \frac{\hat{Q}_1(t)}{[\hat{F}(t)]^\theta} - \int_0^t [\hat{Q}_1(u) - Q_1(u)]\, d\left(\frac{1}{\hat{F}^\theta(u)}\right) + \int_0^t Q_1(u)\, d\left(\frac{1}{\hat{F}^\theta(u)}\right)$$

$$= \frac{\hat{Q}_1(t) - Q_1(t)}{[\hat{F}(u)]^\theta} - \int_0^t [\hat{Q}_1(u) - Q_1(u)]\, d\left(\frac{1}{\hat{F}^\theta(u)}\right)$$

$$+ \int_0^t \frac{dQ_1(u)}{\hat{F}(u)^\theta} \quad . \tag{3.7}$$

By the dominated convergence theorem

$$\lim_{n\to\infty} \int_0^t \frac{dQ_1(u)}{[\hat{F}(u)]^\theta} = \int_0^t \frac{dQ_1(u)}{[F(u)]^\theta} \quad \text{a.s.,}$$

$$\lim_{n\to\infty} \frac{\hat{Q}_1(t) - \hat{Q}_1(t)}{[\hat{F}\ (u)]^{\theta}} = 0 \text{ a.s.},$$

and

$$\lim_{n\to\infty} \sup \{|\hat{Q}_1(u) - Q_1(u)|\} = 0, \text{ a.s.}$$

Hence, applying the above results to (3.7), the result now follows: //

To obtain bounds on the net survival function based on data from a competing risks experiment, we proceed as follows. First, note that from (3.5) it is true that $\hat{S}_\theta(t)$ is a decreasing function of θ for fixed t. Also, as $\theta \to 1^+$ we have $\hat{S}_\theta(t) \uparrow \exp\left(-\int_0^t \hat{F}^{-1}(u)d\hat{Q}_1(u)\right)$.

which provides an upper bound. Notice that this upper bound corresponds to an assumption of independence. As $\theta \to \infty$ one can show that $\hat{S}_\theta(t) \downarrow \hat{F}(t)$ which corresponds to Peterson's (1976) lower bound.

In practice the above bounds, with $\theta = 1, \infty$, while shorter than Peterson's bounds, may still be quite wide.

Tighter bounds may be obtained by an investigator specifying a range of possible values for θ. If the sample size is sufficiently large and $\theta_1 \le \theta \le \theta_2$, then $\hat{S}_{\theta_2}(t) \le S(t) \le \hat{S}_{\theta_1}(t)$. Specifying θ_1, θ_2

is equivalent to specifying a range of values $\tau_1 < \tau < \tau_2$ for the coefficient of concordance τ since $\theta = (1+\tau)/(1-\tau)$. Hence the primary value of $\hat{S}_\theta(t)$ is in putting bounds on S(t) rather than on estimation of S(t).

IV. EXAMPLE AND COMPARISONS

To illustrate the bounds obtained in the previous section, consider the mortality data reported in Hoel (1972). The data was collected on a group of RFM strain male mice who were subjected to a dose of 300 rads of radiation at age 5-6 weeks. There were three competing risks, thymic lymphoma, reticulutum cell sarcoma, and other causes of death. For illustrative purposes we consider reticulum cell sarcoma as the risk of interest.

Table 1 reports the value of $\hat{S}_\theta(t)$ for concordance $\tau = (\theta - 1)/(\theta + 1)$. The value of $\hat{S}_\theta(t)$ at $\tau = 0$ corresponds to Nelson's (1972) hazard rate estimator assuming independence. Peterson's upper and lower bound ($\tau = 1$) are also reported as are Fisher and Kanerek's bounds and the Slud and Rubenstein bounds for several values which reflect a positive association between risks.

From Table 1 we first note that Peterson's bounds are very wide. Substantial improvement is obtained if one assumes a non-negative dependence structure between risks (See Table 2). Further tightening of these bounds is achieved by assuming that τ is in the range 0 to .5 where the width of the boundaries is at most about 50% of that of Peterson's bounds.

Substantial improvement in the general bounds is also obtained by the bounds of Fisher and Kanerek or Slud and Rubenstein. The bounds of Fisher and Kanerek assume a specific censoring pattern and require a specification of a stretching constant α. Without some additional information, such specification may be impossible. Slud and Rubenstein's bounds are for the general dependence structure. Their bounds require the specification of the $\rho(t)$ function. This function is a quantity which is not easily conceptualized by investigators from either a statistical or biological perspective. This makes it questionable whether reasoable upper and lower bounds on $\rho(t)$ can be extracted from one's prior beliefs. The major advantage of the bounds printed in this paper is that they require only the specification of an upper and lower concordance, a measure quite familiar to most investigators and easily explainable to nonstatisticians.

Table 1

BOUNDS ON THE NET SURVIVAL FUNCTION FOR RETICULUM CELL SARCOMA

Time	Peterson's Upper Bound	Proposed Bounds for τ											Fisher and Kanerek for α				Slud and Rubenstein for $\rho(t)$					
		0*	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0**	2.	5.	10.	100.	2	4	6	8	10	20
320	.980	.972	.970	.967	.963	.958	.948	.932	.899	.830	.739	.707	.961	.813	.733	.727	.965	.951	.939	.927	.916	.873
525	.949	.917	.907	.893	.873	.843	.796	.722	.624	.533	.473	.455	.600	.514	.509	.492	.887	.836	.792	.755	.724	.621
600	.848	.708	.666	.619	.562	.498	.436	.385	.351	.332	.322	.322	.393	.367	.361	.357	.608	.485	.419	.383	.363	.340
620	.818	.640	.590	.536	.476	.415	.362	.323	.298	.285	.275	.275	.342	.326	.325	.310	.554	.429	.370	.341	.325	.306
650	.747	.458	.390	.329	.273	.231	.202	.185	.176	.171	.168	.168	.240	.219	.217	.211	.322	.221	.192	.183	.179	.174
675	.707	.343	.271	.216	.175	.147	.132	.123	.119	.117	.117	.117	.179	.163	.162	.162	.213	.142	.128	.124	.123	.122
700	.677	.250	.180	.136	.108	.092	.084	.080	.080	.078	.077	.077	.126	.123	.123	.123	.136	.091	.084	.083	.082	.081
750	.626	.062	.049	.035	.029	.027	.026	.026	.026	.026	.026	.026	.062	.062	.062	.062	.040	.031	.030	.030	.030	.030

* Nelson's Hazard Rate Estimator

** Peterson's Low Bound, Slud-Rubenstein's $\rho(t) = \infty$

Table 2

RELATIVE SIZE OF THE BOUNDS ON NET SURVIVAL
FOR AN ASSUMED DEPENDENCE STRUCTURE
AS COMPARED TO PETERSON'S BOUNDS

Time	$0 \leq \tau \leq 1$	$0 \leq \tau \leq .5$	$0 \leq \tau \leq .7$
350	.9707	.0879	.2674
525	.9352	.2449	.5931
600	.7338	.5171	.6787
620	.6722	.5120	.6298
650	.5009	.4420	.4870
675	.3831	.3576	.3797
700	.2883	.2767	.2833
750	.0600	.0600	.0600

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APPENDIX D

Running Head: Estimators of survival with right-censoring

A COMPARISON OF SEVERAL METHODS OF ESTIMATING THE SURVIVAL FUNCTION WHEN THERE IS EXTREME RIGHT CENSORING

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ABSTRACT

When there is extremecensoring on the right, the Kaplan-Meier product limit estimator is known to be a biased estimator of the survival function. Several modifications of the Kaplan-Meier estimator are examined and compared with respect to bias and mean squared error.

Key words: Adjusted Kaplan-Meier survival estimation,
Bias of survival function, Life-testing,
Survival analysis, Right censoring

1. Introduction

In human and animal survival studies, as well as in life-testing experiments in the physical sciences, one method of estimating the underlying survival distribution (or the reliability of a piece of equipment) which has received widespread attention has been the Kaplan-Meier product limit estimator (Kaplan and Meier, 1958).

For the situation in which the longest time an individual is in a study (or on test) is not a failure time, but rather a censored observation, it is well known that there are many complex problems associated with any statistical analysis (Lagakos, 1979). In particular, the Kaplan-Meier product limit estimator *is biased on the low side* (Gross and Clark, 1975). In the case of many censored observations larger than the largest observed failure time this bias tends to be worse. Estimated mean survival time and selected percentiles, as well as other quantities dependent on knowledge of the tail of the survival function, will also exhibit such biases.

A practical situation which motivates this study is a large-scale animal experiment conducted at the National Center for Toxicological Research (NCTR) where mice were fed a particular dose of a carcinogen. The goals of this experiment were to assess the effects of the carcinogen on survival and on age-specific tumor incidence. Towards this end, mice were randomly divided into three groups and followed until death or until a prespecified group censoring time (280, 420, or 560 days) was reached at which time all those still alive in a given group were sacrificed. Often there were many surviving mice in all three groups at the group's sacrifice time.

In general, we consider an experiment with n individuals under study and censoring is permitted. Let $t_{(1)}, \ldots, t_{(m)}$ denote the m ordered failure times of those m individuals whose failure time is actually observed ($t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(m)}$). The remaining n-m individuals have been censored at various points in time. It will be useful to introduce the notation S_j = number of survivors just prior to time $t_{(j)}$, i.e., S_j is the number of individuals still under observation at the time $t_{(j)}$, including the one that died at $t_{(j)}$. Then the Kaplan-Meier product limit estimator (assuming no ties among the $t_{(j)}$'s) of the underlying survival function, $\bar{P}(t) = P(T>t)$, is

$$\hat{\bar{P}}(t) = \begin{cases} 1 & \text{for} \quad t<t_{(1)} \\ \prod_{\ell=1}^{j} (S_\ell - 1)/S_\ell & \text{for} \quad t_{(j)} \leq t<t_{(j+1)} \\ 0 & \text{for} \quad t \geq t_{(m+1)} \end{cases} \tag{1}$$

for $j=1,...,m$, where $t_{(m+1)}=t_c$ if the longest time an individual is on study is a censoring time or $t_{(m+1)}=\infty$ if the longest time an individual is on study is a death.

This paper, first, proposes in Section 2 some methods of "completing" the Kaplan-Meier estimator of the survival function by i) replacing those censored observations that are larger than the last observed failure time by their expected order statistics, ii) using a Weibull distribution to estimate the tail probability, $\bar{P}(t)$, $t>t_c$, and iii) employing a method suggested by Brown, Hollander and Korwar (BHK) (1974). The second purpose is to demonstrate the magnitude of the bias and mean squared error (MSE) of the Kaplan-Meier estimator and to compare all methods of "completing" $\hat{\bar{P}}(t)$, in the context of the aforementioned mouse study, utilizing simulated lifetimes from exponential, Weibull, lognormal, and bathtub-shaped hazard function distributions. These results are presented in Section 3.

2. Completion of Kaplan-Meier Product-Limit Estimator

2.1 Expected Order Statistics

One method of attempting to "complete" $\bar{P}(t),t>t_c$, would be to "estimate" the failure times for those censored observations which are larger than the longest observed lifetime. Let n_c be the number of censored observations larger than $t_{(m)}$. A theorem regarding the conditional distributions of order statistics states that for a random sample of size n from a continuous parent, the conditional distribution of

$T_{(u)}$, given $T_{(n-n_c)} = t_{(n-n_c)}$, $u>n-n_c$, is just the distribution of the $(u-n + n_c)$th order statistic in a sample of size n_c drawn from the parent distribution truncated on the left at $t = t_{(n-n_c)}$ (David 1981, p. 20).

For computational purposes, take t_c as an estimate of the $(n-n_c)$th order statistic. Then find the expected value of the n_c order statistics from the parent distribution truncated on the left at t_c. Since the Weibull distribution with survival function $\bar{P}(t) = \exp(-t^k/\theta)$ has been widely accepted as providing a reasonable fit for lifetime data, it seems reasonable to employ the results of Weibull distribution theory to complete $\bar{P}(t)$, $t>t_c$. (It should be noted that any distribution which is reasonable for the specific situation may be used.) The expected values of Weibull order statistics up to sample size 40 for location parameter = 1 and shape parameter = 0.5 (0.5)4(1)8 may be found in Harter (1969). For larger sample sizes he states a recurrence relation which may be used.

To compute expected values of the n_c order statistics in question, values for k and θ must be chosen. One approach is to use the maximum likelihood estimators, $\hat{k}$ and $\hat{\theta}$, computed by using all observations to estimate k and θ. A second approach, due to White (1969), uses least squares estimates of k and θ obtained by fitting the model

$$\ln (t_{(j)})=(1/k)\ln\theta+(1/k)\ln H(t_{(j)}) \qquad (2)$$

to the $t_{(j)}$'s where $H(t_{(j)})$ is the estimated cumulative hazard rate at $t_{(j)}$ obtained from the Kaplan-Meier estimator. Based on our Monte Carlo study we found the maximum likelihood estimators performed better in all cases than did the least squares estimators. Consequently, the method of least squares will be dropped from future discussion in this paper.

The survival function for a Weibull random variable, truncated on the left at t_c, is

$$\bar{P}_T(t) = \exp\{-(t^k - t_c^k)/\theta\}\ ,\ t>t_c. \qquad (3)$$

So, by the theorem on order statistics stated at the beginning of this section, the conditional distribution of $T_{(u)}$, given $T_{(n-n_c)}=t_{(n-n_c)}$ $(u=n-n_c+1,\dots,n)$ will be approximated by the $(u-n+n_c)$th order statistic in a sample of n_c drawn from (3). For simplicity, let $j = u - n + n_c$, so that $j = 1,\dots,n_c$.

Now the expected value of the jth order statistic from (3) is

$$E(T_{j:n_c}) = n_c\binom{n_c-1}{j-1}\int_{t_c}^{\infty} t\{P_T(t)\}^{j-1}\{\bar{P}_T(t)\}^{n_c-j+1}(kt^{k-1}/\theta)dt$$

$$=n_c\binom{n_c-1}{j-1}\int_0^{\infty}(y^k+t_c^k)^{1/k}\{P(y)\}^{j-1}\{\bar{P}(y)\}^{n_c-j+1}(ky^{k-1}/\theta)dy \quad (4)$$

where $\bar{P}(y) = \exp(-y^k/\theta), y=(t^k-t_c^k)^{1/k}\geq 0$.

and $T_{j:n_c}$ is the jth order statistic in a sample of size n_c.

Equation (4) can also be written as

$$E(T_{j:n_c})=n_c\binom{n_c-1}{j-1}\int_0^{\infty}(\theta z^k+t_c^k)^{1/k}\{P(z)\}^{j-1}\{\bar{P}(z)\}^{n_c-j+1}kz^{k-1}dz \quad (5)$$

where $\bar{P}(z) = \exp(-z^k)$, $z = (y/\theta)^{1/k} \geq 0$.

Now $E(T_{j:n_c})$ may be crudely estimated by

$$\{\hat{\theta}\{E(Z_{j:n_c})\}^{\hat{k}} + t_c^{\hat{k}}\}^{1/\hat{k}} \quad (6)$$

where $E(Z_{j:n_c})$ is the expected value of the jth order statistic from a sample of size n_c determined from Harter's (1969) tables or recurrence relation and $\hat{\theta}$ and $\hat{k}$ are maximum likelihood estimators of θ and k respectively.

These n_c estimated expected order statistics may then be treated as "observed" lifetimes in adjusting (or "completing") the estimated survival function computed in (1). The area under the estimated survival function up to t_c remains unchanged. The area under the extended estimated survival function based on the n_c estimated expected order statistics is then added to the initial area to get a more precise estimate of $\bar{P}(t)$ (EOS extension).

2.2 <u>Weibull Maximum Likelihood Techniques</u>

A straightforward approach to completing $\hat{\bar{P}}(t)$ is to set

$$\hat{\bar{P}}(t) = \exp(-t^k/\theta) \qquad \text{for } t>t_c. \quad (7)$$

Estimates of k and θ based on all observations can be obtained by either the maximum likelihood (WTAIL) or the least squares method. However, our study found the completion using maximum likelihood estimators was always better in terms of bias and mean squared error.

One ostensible suggestion for improvement of this estimator would be to "tie" the estimated tail to the product-limit estimator at t_c. Two methods were attempted to accomplish this goal. First, the likelihood was maximized with respect to k and θ subject to the constraint that $\exp(-t_c^k/\theta)=\bar{\hat{P}}(t_c)$. This method will be referred to as the restricted MLE tail probability estimate (RWTAIL extension). Second, a scale-shift was performed on the tail probability in (7) so as to tie it to the product-limit estimator. This method led to higher biases and mean squared errors of the survival function and will be dropped from further discussion in this paper.

2.3 <u>BHK Type Methods</u>.

The Brown-Hollander-Korwar completion of the product-limit estimator sets

$$\bar{\hat{P}}(t) = \exp(-t/\theta^*) \text{ for } t>t_c \tag{8}$$

where θ^* satisfies

$$\bar{\hat{P}}(t_c) = \exp(-t_c/\theta^*).$$

In the BHK spirit we tried to complete $\bar{P}(t)$ by a Weibull function which used estimates of k and θ, k^* and θ^*, that satisfied the following two relations

$$\bar{\hat{P}}(t_{(m)}) = \exp(-t_{(m)}^{k^*}/\theta^*)$$

and

$$\bar{P}\hat{}(t_{(m-1)}) = \exp(-t_{(m-1)}^{k*}/\theta^*).$$

The latter method also led to consistently poor performance and the results will not be presented.

3. A Comparison of the Various Methods

A simulation study of data like that collected at NCTR was performed. Three groups of 48 lifetimes were simulated with all testing stopping at 280, 420, and 560 days for the three groups, respectively. Distributions with mean survival times of 400, 500, and 600 days were used. The generated lifetimes greater than or equal to the sacrifice time for that particular group were considered as censored. The remaining set of observed lifetimes, along with the number censored at the three sacrifice times constituted a single sample. For each of the distributions studied, 1000 such samples were generated. Weibull distributions with shape parameters .5, decreasing failure rate , 1, constant failure rate, and 4, increasing failure rate, were used. Lognormal distributions, failure rate changes from increasing to decreasing, with first two moments comparable to the above Weibull distributions with k=1 and k=4 were also used. Finally, a bathtub hazard model, of Glaser (1980), failure rate changes from decreasing to increasing, was used. This distribution is a mixture of an exponential of parameter λ with probability 1-p and a gamma with parameter λ and index of 3 with probability p. Mixing parameters of p=.1 and p=.4 were used.

The bias and MSE for the estimation of the tail probabilities, i.e., the completed portion of the product-limit estimator, were calculated for each hypothesized distribution and for each competing method of completion. Since these results were extremely similar to those found in estimating mean survival time, $\hat{\mu}=\int_0^\infty \hat{\bar{P}}(t)\,dt$, we only show the bias and MSE of each competing estimator of μ in Table 1. This also allows us to demonstrate the magnitude of the bias and MSE of the product-limit estimator of μ. The bias and MSE for estimating the 90th percentile are also presented for the various estimation methods in Table 2. As one would expect, the Kaplan-Meier (K-M) estimator performs considerably more poorly than the other estimation schemes. The BHK extension does very well if the underlying distribution is exponential or lognormal with first two moments compatible with the exponential. BHK does reasonably well for the bathtub shaped hazard model but it performs very poorly for the Weibull with increasing failure rate and for the lognormal with first two moments compatible with the Weibull.

The remaining three extensions (EOS, WTAIL, and RWTAIL) appear to be somewhat comparable. Each of them are best under certain circumstances although many times the biases and MSE's are so close to one another that they are essentially equivalent. Only the EOS extension has the desirable property of never being worst. It usually is competitive with the method that is best.

Ordering the extensions from the standpoint of simplicity, simplest to most complex, we have BHK, WTAIL, RWTAIL, and EOS.

In summary, the Kaplan-Meier estimator should probably be extended in the presence of extreme right censoring. One's choice of extension depends upon one's knowledge of the distribution of lifetimes under consideration and the extent of computer facilities available. If the data follow an exponential type distribution or if no computer facilities are present the BHK method is the extension of choice due to its simplicity. If the data exhibit a non-constant failure rate and computer facilities are available then the RWTAIL or EOS extensions seem to be advisable to use.

ACKNOWLEDGMENTS

This work was supported in part by the Air Force Office of Scientific Research under Contract AFOSR-82-0307.

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TABLE 1

Bias/100(and MSE/$(100)^2$) for Estimating Mean Survival Time for Various Methods of Completion

Distributions	μ	Mean % censored at 560 days	K-M	BHK Extension	Estimated Order Statistic Extension	Weibull WTail Extension	Restricted Weibull RWTAIL Extension
Weibull	400	18.7	-2.000[w]	-1.462	-.101[b]	.131	.206
			(4.034)[w]	(2.271)	(1.172)	(1.160)[b]	(1.543)
k=.5	500	22.3	-2.802[w]	-2.078	-.176[b]	.208	.299
			(7.886)[w]	(4.498)	(1.922)[b]	(2.344)	(3.292)
	600	25.5	-3.625[w]	-2.704	-.187[b]	.344	.479
			(13.179)[w]	(7.522)	(3.025)[b]	(4.275)	(6.031)
	400	24.6	-.991[w]	-.047	-.046	.016[b]	.0379
			(1.011)[w]	(.215)[b]	(.257)	(.275)	(.343)
k=1	500	32.6	-1.632[w]	-.049	-.047[b]	.073	.116
			(2.696)[w]	(.416)[b]	(.535)	(.508)	(.705)
	600	39.3	-2.359[w]	.022[b]	.034	.140	.214
			(5.592)[w]	(.596)[b]	(.987)	(1.023)	(1.353)
	400	27.5	-.036	.136[w]	-.005	.003[b]	.004
			(.012)[b]	(.053)[w]	(.013)	(.014)	(.014)
k=4	500	34.6	-.314	1.507[w]	-.020	.014[b]	.019
			(.109)	(2.830)[w]	(.036)[b]	(.041)	(.044)
	600	59.9	-.903	5.982[w]	.144	.028[b]	.039
			(.822)	(41.430)[w]	(4.168)	(.147)[b]	(.157)
lognormal	400	20.6	-.868[w]	-.178[b]	-.544	-.586	-.412
			(.777)[w]	(.179)[b]	(.363)	(.403)	(.267)
(k=1)	500	29.0	-1.427[w]	-.150[b]	-.865	-.918	-.696
			(2.060)[w]	(.323)[b]	(.855)	(.938)	(.644)
	600	36.9	-2.079[w]	-.022[b]	-1.234	-1.281	-1.038
			(4.345)[w]	(.571)[b]	(1.679)	(1.800)	(1.301)
	400	8.6	-.070	.129[w]	-.047	-.053	-.027[b]
			(.014)[b]	(.056)[w]	(.014)[b]	(.014)[b]	(.014)[b]
(k=4)	500	29.1	-.330	1.033[w]	-.170	.181	-.135[b]
			(.118)	(1.459)[w]	(.051)	(.055)	(.043)[b]
	600	54.5	-.853	4.430[w]	-.391	-.392	-.356[b]
			(.734)	(23.159)[w]	(.199)	(.199)[b]	(.177)[b]
Bathtub	400	18.6	-1.069	-.185	-.170	1.125[w]	.063[b]
			(1.175)	(.234)[b]	(.260)	(1.745)[w]	(.361)
p=.1	500	26.1	-1.722[w]	-.259	-.202	1.523	.046[b]
			(2.996)	(.427)[b]	(.560)	(3.230)[w]	(.608)
	600	32.6	-2.452[w]	-.362	-.310	1.761	.047[b]
			(6.043)[w]	(.727)[b]	(.982)	(4.490)	(1.254)
	400	8.1	-1.786[w]	-1.543	-1.547	-.936	.343[b]
			(3.218)[w]	(2.463)	(2.476)	(1.081)	(.544)[b]
p=.4	500	13.3	-2.370[w]	-1.826	-1.814	-.825	.585[b]
			(5.649)[w]	(3.472)	(3.446)	(1.031)[b]	(1.303)
	600	18.7	-3.072[w]	-2.191	-2.175	-.875	.841[b]
			(9.466)[w]	(5.013)	(4.983)	(1.285)[b]	(2.792)

[b] Best estimation method

[w] Worst estimation method

TABLE 2

Bias/100(and MSE/$(100)^2$) for Estimating 90th Percentile for Various Methods of Completion

Distributions	μ	K-M	BHK Extension	Estimated Order Statistic Extension	Weibull WTAIL Extension	Restricted Weibull RWTAIL Extension
Weibull	400	-5.017[w]	-2.858	1.691	.234[b]	.458
		(25.185)[w]	(9.358)	(16.424)	(7.524)[b]	(10.812)
k=.5	500	-7.655[w]	-4.620	1.897	.418[b]	.642
		(58.604)[w]	(22.711)	(24.276)	(14.319)[b]	(21.442)
	600	-10.306[w]	-6.390	2.213	.734[b]	1.064
		(106.21)[w]	(42.449)	(36.895)	(25.419)[b]	(37.911)
	400	-3.610[w]	.064[b]	.248	.084	.067
		(13.035)[w]	(1.892)[b]	(2.423)	(1.980)	(2.945)
k=1	500	-5.913[w]	.096[b]	.289	.121	.306
		(34.963)[w]	(2.995)[b]	(4.681)	(4.361)	(5.903)
	600	-8.216[w]	.244[b]	.610	.418	.550
		(67.459)[w]	(4.198)[b]	(9.247)	(8.331)	(10.792)
	400	-.045	.098[w]	-.007[b]	-.037	-.011
		(.038)[b]	(.236)[w]	(.060)	(.047)	(.063)
k=4	500	-1.195	5.324[w]	-.031	-.026	.024[b]
		(1.429)	(33.091)[w]	(.146)	(.141)[b]	(.177)
	600	-2.554	17.913[w]	.120	.090	.068[b]
		(6.524)	(355.02)[w]	(.794)	(.676)	(.641)[b]
lognormal	400	-2.628[w]	-.044[b]	-1.263	-1.758	-.967
		(6.908)[w]	(1.526)[b]	(1.979)	(3.407)	(1.673)
(k=1)	500	-4.680[w]	.213[b]	-2.354	-2.718	-1.908
		(21.902)[w]	(2.708)[b]	(6.153)	(7.909)	(4.751)
	600	-6.736[w]	.759[b]	-3.507	-3.766	-2.980
		(45.373)[w]	(4.764)[b]	(13.123)	(14.981)	(10.257)
	400	-.085	.161	-.038	-.162[w]	-.024[b]
		(.060)[b]	(.409)[w]	(.081)	(.065)	(.093)
(k=4)	500	-1.251	3.722[w]	-.584	-.657	-.484[b]
		(1.566)	(17.654)[w]	(.403)	(.495)	(.318)[b]
	600	-2.621	13.695[w]	-1.214	-1.236	-1.158[b]
		(6.872)	(210.30)[w]	(1.616)	(1.662)	(1.498)[b]
Bathtub	400	-3.629[w]	-.177	.053[b]	-.104	.105
		(13.167)[w]	(1.717)[b]	(2.052)	(2.058)	(3.190)
p=.1	500	-6.068[w]	-.457	-.071	-.208	.004[b]
		(36.826)[w]	(2.955)[b]	(4.702)	(3.619)	(5.245)
	600	-7.997[w]	-.318	.043	-.244	-.014[b]
		(63.954)[w]	(4.330)[b]	(7.786)	(7.608)	(9.923)
	400	-.347	.143[b]	.276	1.154[w]	.981
		(.273)[b]	(.844)	(1.078)	(3.877)	(4.747)[w]
p=.4	500	-1.425	.521[b]	.764	1.699	1.718[w]
		(2.035)	(1.540)[b]	(2.067)	(8.574)	(10.714)[w]
	600	-3.554[w]	-.137	.132[b]	2.304	2.450
		(12.628)[w]	(1.804)[b]	(2.352)	(17.530)	(22.456)

[b] Best estimation method

[w] Worst estimation method

Appendix E

ESTIMATING RELIABILITY FOR BIVARIATE

EXPONENTIAL DISTRIBUTIONS

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30 September, 1984

Annual Report for Period 1 October 1983 - 30 September 1984

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ESTIMATING RELIABILITY FOR BIVARIATE EXPONENTIAL DISTRIBUTIONS

by

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ABSTRACT

The problem of estimating reliability for the bivariate exponential distributions of Block and Basu (1974) and Marshall and Olkin (1967) is considered. For Bock and Basu's model a minimum variance unbiased estimator of the joint survival function is obtained in the case of identically distributed marginals. For the non-identically distributed case the performance of the maximum likelihood estimator and the jackknifed maximum likelihood estimator is studied. For Marshall and Olkin's model the performance of several different parameter estimators and bias reduction techniques for estimating joint reliability are considered.

KEY WORDS: Minimum Variance Unbiased Estimators; Bivariate Exponential; Reliability; Maximum Likelihood Estimator; Jackknife; Survival Function.

AUTHORS' FOOTNOTE

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1. INTRODUCTION

Let X, Y have either the bivariate exponential distribution (BVE) of Marshall and Olkin (1967) or the absolutely continuous bivariate exponential distribution (ACBVE) of Block and Basu (1974). These two distributions have found considerable use as models for both physical and biological systems. The problem of interest is to estimate the joint reliability function, $\bar{F}(x,y) = P(X>x, Y>y)$, for each of these two distributions. A natural estimator of $\bar{F}(x,y)$ is obtained by substituting in the appropriate expression for $\bar{F}(x,y)$ good estimators of the model parameters. Often, as seen in Pugh (1963), Basu (1964) or Basu and El Mawaziny (1978), such estimators can be considerably biased. We wish to obtain reduced biased estimators of $\bar{F}(x,y)$ for both the BVE and ACBVE distributions.

In Section 2 this estimation problem is considered for the ACBVE. In the case of identically distributed marginals, using the Roa-Blackwell and the Lehmann-Scheffé theorems we obtain minimum variance unbiased estimators (UMVUE) of $\bar{F}(x,y)$. In the case of non-identically distributed marginals this approach fails since there are no complete sufficient statistics. Here we investigate the performance of the maximum likelihood estimator as well as the jacknifed maximum likelihood estimator.

In Section 3 we consider the estimation of $\bar{F}(x,y)$ for the BVE. Again there are no complete sufficient statistics so no minimum variance unbiased estimators can be obtained. Several different methods for estimating parameters are considered. For each estimation procedure we consider several bias reduction techniques.

2. ABSOLUTELY CONTINUOUS BIVARIATE EXPONENTIAL

2.1 Introduction

Let (X, Y) have the absolutely continuous bivariate exponential distribution of Block and Basu (1974) with parameters $\lambda_1, \lambda_2 > 0, \lambda_{12} \geq 0$ ($(X, Y) \sim$ ACBVE $(\lambda_1, \lambda_2, \lambda_{12})$). This distribution is closely related to the bivariate exponential of Freund (1961). It has been used by Gross, Clark and Lui (1971) and Gross (1973) to model the lifetimes of two organ systems and by Gross and Lam (1981) for modeling paired survival time data such as survival of a tumor remission when a patient receives two types of treatment.

For this model the joint reliability function is

$$\bar{F}(x,y) = \frac{\lambda}{(\lambda_1+\lambda_2)} \exp(-\lambda_1 x-\lambda_2 y-\lambda_{12} \max(x,y))$$

$$- \frac{\lambda_{12}}{(\lambda_1+\lambda_2)} \exp(-\lambda \max(x,y)), \text{ for } x,\ y > 0,$$

$$\text{with} \quad \lambda = \lambda_1 + \lambda_2 + \lambda_{12}. \qquad (2.1.1)$$

This distribution has the bivariate loss of memory property (LMP) defined by Block and Basu (1974). It is the absolutely continuous part of the Marshall and Olkin (1967) bivariate exponential.

We shall consider two cases for estimating $\bar{F}(x,y)$, one where the marginals are identically distributed and the general model.

2.2 Equal Marginals

Consider the model (2.1.1) with $\lambda_1 = \lambda_2 = \alpha$ and $\lambda_{12} = \beta$. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be a random sample from (2.1.1). Let $U_1 = \Sigma\max(x_i, y_i)$ and $U_2 = \Sigma(x_i + y_i)$. Mehrotra and Michalek (1976) show that (U_1, U_2) is a complete sufficient statistic for (α,β). The MLE of α,β are given by

$$\hat{\alpha} = n\left(\frac{1}{u_2-u_1} - \frac{1}{2u_1-u_2}\right),\ \hat{\beta} = n\left(\frac{2}{2u_1-u_2} - \frac{1}{u_2-u_1}\right). \qquad (2.2.1)$$

These estimators are biased by a factor of $n/(n-1)$ so the estimators $\tilde{\alpha} = \frac{n-1}{n}\hat{\alpha}$ and $\tilde{\beta} = \frac{n-1}{n}\hat{\beta}$ are the UMVUE of α and β. Two natural estimators of $\bar{F}(x,y)$ are obtained by sustituting either $(\hat{\alpha}, \hat{\beta})$ or $(\tilde{\alpha}, \tilde{\beta})$ in (2.1.1).

We now use the method proposed by Basu (1964) to obtain the UMVUE of $\bar{F}(x,y)$.

Define

$$\phi(x,y:\ X,Y) = \begin{cases} 1 & \text{if} \quad X > x,\ Y > y \\ 0 & \text{otherwise} \end{cases} \qquad (2.2.2)$$

Clearly $\phi(x,y:\ X,Y)$ is an unbiased estimator of $\bar{F}(x,y)$ based on a random sample of size one from a ACBVE (α,α,β). By the Rao-Blackwell and Lehmann-Scheffé theorems the estimator $\tilde{\bar{F}}(x,y) = E(\phi(x,y;X,Y)|u_1,u_2)$ is the UMVUE of $\bar{F}(x,y)$.

To simplify the calculations let $T = U_2-U_1$ and $V = 2U_1-U_2$, that is $T = \Sigma \min(X_i,Y_i)$ and $V = \Sigma \max(X_i,Y_i) - \Sigma \min(X_i,Y_i)$. From Mehrotra and Michalek (1976), the joing density of (T,V) is

$$f(t,v) = \begin{cases} \dfrac{(2\alpha+\beta)^n(\alpha+\beta)^n}{[(n-1)!]^2}\, t^{n-1}v^{n-1}\exp(-(2\alpha+\beta)t-(\alpha+\beta)v), & t,\ v > 0 \\ 0 & \text{otherwise.} \end{cases} \qquad (2.2.3)$$

Now split the sample of size n into two independent sub-samples of sizes one and n-1, respectively. Let (Z_1, Z_2) denote the sample of size one and let T_1, V_1 denote the statistics T and V defined on the remaining n-1 observations. The joint density of (Z_1, Z_2, T_1, V_1) is

$$f(z_1,z_2,t_1,v_1)=\begin{cases} \dfrac{(2\alpha+\beta)^n(\alpha+\beta)^n}{2[(n-2)!]^2} t_1^{n-2}v_1^{n-2}\exp[-(2\alpha+\beta)t_1-(\alpha+\beta)v_1- \\ \qquad \alpha z_1-(\alpha+\beta)z_2] \text{ if } z_1 < z_2 \\ \dfrac{(2\alpha+\beta)^n(\alpha+\beta)^n}{2[(n-2)!]^2} t_1^{n-2}v_1^{n-2}\exp[-(2\alpha+\beta)t_1-(\alpha+\beta)v_1- \\ \qquad (\alpha+\beta)z_1-\alpha z_2] \text{ if } z_2 < z_1 \end{cases} \quad (2.2.4)$$

for $t_1, v_1 > 0$.

Clearly $V = V_1 + \max(Z_1, Z_2) - \min(Z_1, Z_2)$ and $T = T_1 + \min(Z_1, Z_2)$. Hence the joint density of (Z_1, Z_2, T, V) is

$f(z_1,z_2,t,v) =$

$$\frac{(2\alpha+\beta)^n(\alpha+\beta)^n}{2[(n-1)!]^2} (t-z_1)^{n-2}(v-z_2+z_1)^{n-2}\exp(-(2\alpha+\beta)t-(\alpha+\beta)v) \text{ for}$$

$$0 < v,\ t;\ 0 < z_1 < t;\ z_1 < z_2 < v + z_1$$

$$\frac{(2\alpha+\beta)^n(\alpha+\beta)^n}{2[(n-1)!]^2} (t-z_2)^{n-2}(v-z_1+z_2)\exp(-2\alpha+\beta)t-(\alpha+\beta)v)$$

$$0 < v,t;\ \ 0 < z_2 < t\ ;z_2 < z_1 < v + z_2$$

(2.2 5)

Thus the conditional distribution of Z_1, Z_2 given T, V is

$$f(z_1,z_2|T=t, V=v)=\begin{cases} \dfrac{(n-1)^2}{2}\dfrac{(t-z_1)^{n-2}(v-z_2+z_1)^{n-2}}{t^{n-1}\ v^{n-1}}, 0 < z_1 < t, \\ \qquad z_1 < z_2 < v + z_1 \\ \dfrac{(n-1)^2}{2}\dfrac{(t-z_2)^{n-2}(v-z_1+z_2)^{n-2}}{t^{n-1}\ v^{n-1}}, 0 < z_2 < t, \\ \qquad z_2 < z_1 < v + z_2 \end{cases}$$

(2.26)

To find $E(\phi(x,y;\ Z_1,Z_2)|T = t,\ V = v)$ consider three cases.

Case 1: $t > x = y > 0$. Here,

$$E(\phi(x,y;Z_1,Z_2)|T=t,\ V=v) =$$

$$= \quad 2\int_x^t \int_{z_1}^{v+z_1} \frac{(n-1)^2(t-z_1)^{n-2}(v-z_2+z_1)^{n-2}}{2\,t^{n-1}\,v^{n-1}}\,dz_2dz_1$$

$$= \quad (1-\frac{x}{t})^{n-1} \tag{2.2.7}$$

Case 2: $x < y < t$. Here,

$$E(\phi(x,y;Z_1,Z_2)|T=t,\ V=v) = \iint_{\{(z_1,z_2):z_1>y,z_2>y\}} f(z_1,z_2|t,v)dz_1dz_2 +$$

$$+ \iint_{\{(z_1,z_2):\ x<z_1<y,\ y<z_2\}} f(z_1,\ z_2|t,v)dz_1dz_2 = (1-\frac{y}{t})^{n-1} +$$

$$+ \int_x^y\int_y^{v+z_1} \frac{1}{2}\,\frac{(n-1)^2(t-z_1)^{n-2}(v+z_1-z_2)^{n-2}}{t^{n-1}\,v^{n-1}}\,dz_2dz_1 = (1-\frac{y}{t})^{n-1} +$$

$$+ \frac{(n-1)}{2v^{n-1}t^{n-1}} \sum_{k=0}^{n-1} (-1)^k \frac{\binom{n-1}{k}(v-y+t)^{n-1-k}}{(n+k-1)} [(t-x)^{n+k-1}-(t-y)^{n+k-1}] \tag{2.2.8}$$

Case 3: $t > x > y > 0$.

By symmetry $E(\phi(x,y;\ Z_1,Z_2)|T=t,\ V=v) = (1-\frac{x}{t})^{n-1} +$

$$\frac{(n-1)}{2v^{n-1}t^{n-1}} \sum_{k=0}^{n-1} \frac{(-1)^k\binom{n-1}{k}}{(n+k-1)}(v-x+t)^{n-1-k}[(t-y)^{n+k-1}-(t-x)^{n+k-1}]. \tag{2.2.9}$$

2.3 Unequal Marginals

When (X,Y) is ACBVE $(\lambda_1,\lambda_2, \lambda_{12})$ with λ_1 not known to be equal to λ_2, there does not exist a set of complete, sufficient statistics for $(\lambda_1,\lambda_2,\lambda_{12})$. Hence, the technique described in section 2.2 fails. Maximum likelihood estimators of $(\lambda_1,\lambda_2,\lambda_{12})$ are obtained numerically by maximizing the likelihood equation as described in Block and Basu (1974). The maximum likelihood estimator, $\hat{\bar{F}}(x,y)$ of $\bar{F}(x,y)$ is obtained by substituting these estimators into (2.1.1).

For small sample sizes, this estimator may be highly biased. To reduce this bias we consider the jackknifed version of the MLE estimator. This estimator is constructed as follows: Let $\hat{\bar{F}}^{(j)}_{(n-1)}(x,y)$ be the MLE of $\bar{F}(x,y)$ based on the subsample of size n-1 obtained by deleting the j^{th} observation from the original sample. The jackknifed version of $\hat{\bar{F}}(x,y)$ is

$$\hat{\bar{F}}_{jack}(x,6) = n\,\hat{\bar{F}}(x,y) - \frac{(n-1)}{n}\sum_{j=1}^{n}\hat{\bar{F}}^{(j)}_{(n-1)}(x,y). \qquad (2.3.1)$$

Miller (1974) shows that this estimator removes the n^{-1}th order term in the expansion of the bias of $\bar{F}(x,y)$.

To study the performance of the MLE and the jackknifed MLE of $\bar{F}(x,y)$, a simulation study was performed. For various values of $\lambda_1,\lambda_2,\lambda_{12}$ and n, 500 ACBVE samples were generated by the method of Friday and Patil (1977). Values of (x,y) were picked so that $\bar{F}(x,y) = .9$. The study showed that the jackknifed maximum likelihood estimator had significantly smaller bias than the MLE. For sample sizes of 10 or larger, the bias of this estimator is not statistically different from zero. However, the jackknifed MLE has a slightly larger mean squared error than the MLE in all cases considered.

3. BIVARIATE EXPONENTIAL

3.1 Parameter Estimation

We say (X,Y) follows the bivariate exponential distribution of Marshall and Olkin (1967) with parameters $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_{12} \geq 0$((X,Y) is BVE $(\lambda_1, \lambda_2, \lambda_{12})$) if the joint survival function is

$$P(X > x, Y > y) = \overline{F}(x,y) = \exp(-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x,y)) \quad (3.1.1)$$

for $x > y > 0$. This distribution is not absolutely continuous since $P(X = y) = \lambda_{12}/(\lambda_1 + \lambda_2 + \lambda_{12})$. The marginals are exponential as is min(X,Y). This is the only bivariate distribution with exponential marginals and the loss of memory property (LMP) as defined in Block and Basu (1974).

To estimate λ_1, λ_2, λ_{12} based on a random sample (X_1, Y_1), ... (X_n, Y_n), let n_1, n_2, n_{12} be the number of observations with X_i less, greater, and equal to Y_i, respectively. Let $T = \Sigma\max(X_i, Y_i)$, $S_x = \Sigma X_i$, $S_y = \Sigma Y_i$. Bhattacharyya and Johnson (1971) show that $(n_1, n_2, t_2, s_x, s_y)$ are jointly minimal sufficient but not complete. Hence, the approach of section 2.1 cannot be applied. The maximum likelihood estimators are obtained by numerically maximizing the likelihood equations. Bhattacharyya and Johnson (1971) obtain conditions under which the MLE exist, and show that these estimators are asymptotically trivariate normal with mean $(\lambda_1, \lambda_2, \lambda_{12})$.

Bemis, Bain, and Higgins (1972) have obtained method of moments estimators of the parameters. Proschan and Sullo (1976) obtained estimators of the parameters by using a first iterate in the likelihood equations. Arnold (1968) gives estimators of λ_i based on n_1, n_2, n_{12} and

$U = \Sigma \min(X_i, Y_i)$. In the competing risks framework where only the minimum of X and Y is observed, these estimators are the unique minimum variance unbiased estimators of λ_i. All of the above estimators are asymptotically trivariate normal with mean $(\lambda_1, \lambda_2, \lambda_{12})$.

3.2 Estimation of Tail Probability

The problem of interest is to estimate $\overline{F}(x,y)$ given by (3.1.1). A natural method of estimating (3.1.1) is to use one of the above methods to estimate $(\lambda_1, \lambda_2, \lambda_{12})$ and substitute these estimates in (3.1.1).

Several methods may be uwed to reduce the bias of these estimators. The first approach is to expand the substitution estimator in a Taylor series about $(\lambda_1, \lambda_2, \lambda_{12})$ keeping only second order terms. When $E(\hat{\lambda}_i) = \lambda_i$, the bias of the substitution estimator is approximately equal to

$E(\hat{\overline{F}}_{SUB}(x,y)) \cong \overline{F}(x,y)\ [1 + \sigma^2/2]$ where

$$\sigma^2 = (x,y,\ \max(x,y))\Sigma(x,y,\ \max(x,y))' \qquad (3.2.1)$$

and Σ is the appropriate covariance matrix of $(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_{12})$. This suggests a reduced bias estimator of $\overline{F}(x,y)$ given by

$$\hat{\overline{F}}_{TS}(x,y) = \hat{\overline{F}}_{SUB}(x,y)/[1 + \hat{\sigma}^2/2] \qquad (3.2.2)$$

where $\hat{\sigma}^2$ is an estimator of σ^2.

A second approach to the bias of $\hat{\overline{F}}_{SUB}(x,y)$ is through asymptotic theory. Note that $\ln \hat{\overline{F}}_{SUB}(x,y)$ is asymptotically normal with mean $-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)$ and variance σ^2. Thus, for large n, $\hat{\overline{F}}_{SUB}(x,y)$ has a log normal distribution and

$$E(\hat{\bar{F}}_{SUB}(x,y)) \to \bar{F}(x,y)\ \exp(\sigma^2/2) \tag{3.2.3}$$

and $V(\hat{\bar{F}}_{SUB}(x,y)) \to \bar{F}(x,y))^2 e^{\sigma^2}(e^{\sigma^2}-1)$. This suggests a reduced bias estimator of $\bar{F}(x,y)$ given by

$$\hat{\bar{F}}_{LN}(x,y) = \hat{\bar{F}}_{SUB}(x,y)\exp(\hat{\sigma}^2/2). \tag{3.2.4}$$

A third method to reduce the bias of $\bar{F}_{SUB}(x,y)$ is the jackknife as described in Section 2.3.

To compare these estimators, a simulation study was performed. 500 BVE observations were generated for various combinations of n, λ_1, λ_2, λ_{12}. Values of (x,y) were selected so that $\bar{F}(x,y) = .9$.

Several conclusions can be drawn from the study. First, for all bias reduction techniques, those based on Arnold's estimators have a significantly larger mean squared error but a smaller relative bias. Secondly, there appears to be very little difference in the estimators based on the other three methods. For Arnold's estimators, all three bias reduction techniques yield approximately unbiased estimators with comparable mean squared errors. For the other methods, only the jackknifed estimator is approximately unbiased due to bias of the parameters themselves. Our recommendation is to jackknife either the Proschan and Sullo estimator or the method of moments estimator since these are computationally easier than the MLE.

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